APPLIED MATHEMATICS AND STATISTICS LABORATORY

STANFORD UNIVERSITY

CALIFORNIA

CONVEXITY OF FUNCTIONALS BY TRANSPLANTATION

By

G. PÓLYA AND M. SCHIFFER

APPENDIX

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HEINZ HELFENSTEIN

TECHNICAL REPORT NO. 14

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OFFICE OF NAVAL RESEARCH

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G. Pólya - M. Schiffer

Introduction.

In the present paper, we wish to discuss the dependence of various functionals on their domain of definition. The functionals considered will be defined by certain extremum problems or, equivalently, by boundary value problems for partial differential equations of elliptic type. In either case, it is difficult to compute the functional for an arbitrarily given domain, while explicit solutions are available, if at all, only in the case that the domain of definition is of simple or symmetric structure. There arises the question of utilizing the knowledge of the functional for a few special domains in order to obtain information about the same functional in the general case. Two different methods have been applied successfully for this purpose; namely, the method of transplanting extremum functions and the method of variation.

The method of transplantation can be applied if the functional in question is characterized as the extremum value of another functional over a certain function class with respect to the domain of definition D. Suppose that we know the function in D which yields the correct extremum value of the functional; if D′ is another domain referred to D by a one-to-one transformation of a sufficient degree of continuity, we may consider the extremum function of D when properly transplanted as a new function in D′. Here it will not, in general, be the extremum function for the problem considered with respect to D′; but it may still be a competing function for the extremum problem and, in this event, the value it imparts to the D′-functional may be used as an estimate for the actual extremum. Finally, we may

obtain in this way a comparison between the extrema with respect to D and to D′ so that, if the D-extremum is known, a bound for the D′-extremum has been found. The method of transplanting the extremum function has been used systematically in [15] and has provided numerous inequalities and theorems.

The method of variation starts with the assumption that a functional is known for a given domain D and provides a formula for the infinitesimal change of the functional in case the domain D of definition is changed infinitesimally. The variational formula may serve to characterize those domains for which the functional attains extremum values under various side conditions. It happens frequently that the formal structure of the variational formula indicates monotonicity or other growth characteristics of the functional studied. In [6] the fact was stressed that various convexity properties of functionals could be established by considering the second variation of the functional with the domain, and several applications of convexity results were given.

In this paper a certain synthesis between the above two methods is intended. We shall use the method of transplantation within a family of domains D(t) which arise from one given domain D = D(1) by a one-parameter transformation. In this way, we are able to obtain variational formulas and convexity statements for the functionals considered. The information obtained is more extensive than that found by one discrete transplantation of the extremum function while the results available by infinitesimal variation are improved to statements in the large. We consider in this paper only a few simple types of transformation such as conformal transformations in the plane case or one-sided stretchings in the two- and three-dimensional case. But it is easy to apply the same method to other

transformations, as well, and various analogous results can then be obtained. We indicate briefly the partial one-sided stretching of a domain which consists in a stretching of the part of the domain that lies to one side of a given line or plane while the rest of the domain is left unchanged. By using proper partial stretchings, many general variational formulas can be derived by the transplantation argument, a fact which shows the significance of this simple type of transformation.

In Chapter I, we develop our method for functionals such as torsional rigidity, virtual mass, outer conformal radius and electrostatic capacity. Most of these quantities are defined by two complementary extremum problems leading to upper and lower bounds for the functionals. In these cases, our method leads to corresponding complementary convexity statements which are of value in the numerical treatment of the quantity considered.

In Chapter II, we discuss a useful theorem of Poincaré which permits an easy simultaneous estimation of the N first eigenvalues of a general type of eigenvalue problems. This lays the groundwork for the application of our method to eigenvalue problems.

In Chapter III, the convexity of various combinations of eigenvalues is studied in the case in which the domain of definition is deformed either by stretching or by conformal transformation. In the problems treated, the convexity statement is of particular interest, since the eigenvalues considered are defined as minima of certain quadratic functionals and since from the definition itself only upper estimates for the eigenvalues are available in an elementary fashion. Here convexity results lead sometimes to lower bounds and are an important numerical tool. We refer to [10] where the convexity theorem of this paper has been applied in order to find estimates for the membrane eigenvalues of isosceles triangles.

In Chapter IV, we indicate the use which can be made of the fact that the initial domain $D(1)$ has symmetry properties. It is shown that in many cases the derivative of the functional with respect to the parameter t can be found at the point $t = 1$. Various inequalities are derived as illustrative examples.

In Chapter V, we show that the invariance of the class of harmonic functions in D under conformal mapping can be used in order to derive convexity statements for some interesting functionals connected with the Green's function for Laplace's equation. In fact, the invariance of this Green's function under conformal mapping provides the simplest and most frequently used instance of transplantation.

Finally, in the Appendix, a numerical application is given for the torsional rigidity of isosceles triangles and rectangles. This example indicates how far one can go numerically by a proper utilization of a few comparison and convexity theorems.

Chapter I

Transplantation of the Extremal Function

1. Torsional Rigidity.

1.1 We consider a simply-connected, bounded domain D in the (x,y)-plane and define a one-sided stretching of D in the x-direction as follows. We subject the whole plane to the linear transformation

$$(1.1) \qquad x' = tx \qquad y' = y \quad , \qquad t > 0 \quad ,$$

and define D(t) as the image of D under this stretching; we call t the parameter of the stretching and allow for it arbitrary positive values. The original domain D may also be denoted by D(1).

We wish to study the dependence of various functionals of the domains D(t) upon the parameter t and to demonstrate a rather general method which is applicable if the functional in question is defined by an extremum problem. We explain the method for the case of the torsional rigidity P(t) of the domain D(t).

The functional P is defined by means of the stress function f(x,y) for the domain D. This function satisfies the differential equation

$$(1.2) \qquad \Delta f = -2 \quad , \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad ,$$

in D and vanishes on the boundary C of D. In terms of f(x,y), we have [15]:

$$(1.3) \qquad P = \frac{\left(2 \iint\limits_D f \, dx\, dy\right)^2}{\iint\limits_D (\nabla f)^2 dx\, dy} \quad .$$

Using integration by parts and (1.2), we may bring (1.3) also into the alternative forms

$$(1.4) \qquad P = 2 \iint\limits_D f \, dx\, dy = \iint\limits_D (\nabla f)^2 dx\, dy \quad .$$

The advantage of the definition (1.3) over the formulas (1.4) comes from the following theorem: Let F(x,y) be any continuously differentiable

function in D which vanishes on the boundary C; then

$$(1.5) \qquad P \geqq \frac{\left(2\iint\limits_D F\,dx\,dy\right)^2}{\iint\limits_D (\nabla F)^2\,dx\,dy} .$$

Thus, the stress function $f(x,y)$ solves an extremum problem for the ratio on the right side of (1.5) and the torsional rigidity P is the maximum value of this ratio.

Let now $f(x,y)$ be the correct stress function of the domain $D(t_o)$. Then $f(t_o t^{-1}x,y)$ is defined in $D(t)$, vanishes on its boundary $C(t)$ and is an admissible function in the variational problem (1.5) that defines $P(t)$. Therefore,

$$(1.6) \qquad P(t) \geqq \frac{\left(2\iint\limits_{D(t)} f(t_o t^{-1}x,y)\,dx\,dy\right)^2}{\iint\limits_{D(t)} [t_o^2 t^{-2} f_1(t_o t^{-1}x,y)^2 + f_2(t_o t^{-1}x,y)^2]\,dx\,dy}$$

where f_1 and f_2 denote the derivatives of f with respect to its first and second argument, respectively. We change now our variables of integration by referring back to the domain $D(t_o)$; thus, (1.6) becomes after some easy transformations

$$(1.7) \qquad \frac{t}{P(t)} \leq t_o \frac{\tau t_o^2 \iint\limits_{D(t_o)} f_1(x,y)^2\,dx\,dy + \iint\limits_{D(t_o)} f_2(x,y)^2\,dx\,dy}{\left(2\iint\limits_{D(t_o)} f(x,y)\,dx\,dy\right)^2} .$$

Here, we put $\tau = t^{-2}$. Observe that the right side of (1.7) has the form $a\tau + b$ and that for $\tau = \tau_o = t_o^{-2}$ we have equality in (1.7) by virtue of the definition (1.3).

Consider the curve with the ordinate $tP(1)^{-1}$ and the abscissa $\tau = t^{-2}$. As (1.7) shows (since t_o is arbitrary), there passes through each point of this curve a straight line, no point of which lies below the curve. A curve with this property is obviously a convex curve and the straight lines considered are its supporting lines.

We find from (1.7) also the slope of the supporting line at the point τ_o

$$(1.8) \qquad a = \tau_o^{-3/2} \frac{\iint_{D(t_o)} f_1^2 \, dx\, dy}{P(t_o)^2} \geq 0 \quad .$$

Thus, the curve considered in non-decreasing. We collect our results in

Theorem 1. Let $P = P(t)$ be the torsional rigidity of the simply-connected bounded plane domain $D = D(t)$ which arises from $D(1)$ by one-sided stretching in a fixed direction with the parameter t. Then tP^{-1}, considered as a function of the variable $\tau = t^{-2}$, is an increasing function and convex from above.

1.2. We wish to add a few additional remarks. Instead of the expression tP^{-1} we might also consider the functional AP^{-1} where $A(t)$ is the area of the domain $D(t)$. In fact, $A(t)$ is proportional to t and the curve AP^{-1} with the abscissa $\tau = t^{-2}$ has the same properties as the curve tP^{-1}.

If $D(1)$ is a circle with unit radius, $D(t)$ is an ellipse with semi-axes 1 and t. Obviously,

$$(1.9) \qquad f(x,y) = \frac{1-(\tau x^2 + y^2)}{1+\tau} \quad , \qquad \tau = t^{-2} \quad .$$

is the stress function of $D(t)$ and we calculate easily from (1.4)

$$(1.10) \qquad P(t) = \frac{\pi t^3}{1+t^2} \quad , \qquad tP(t)^{-1} = \frac{1}{\pi}(\tau + 1) \quad .$$

Thus, in this particular case the convex curve becomes simply a straight line which shows that the inequality

$$(1.11) \qquad \frac{d^2}{d\tau^2}(tP^{-1}) \leq 0$$

is the best possible one.

1.3. Let us return to the general domain $D(1)$; it contains a certain circle of radius r and is contained in some circle of radius R. We know, therefore, that the domain $D(t)$ contains an ellipse with semi-axes r and rt

and is contained in another ellipse with semi-axes R and Rt. The torsional rigidity P is a monotonic set function as follows immediately from the extremum problem (1.5) which defines it. Hence, using (1.10), we have the estimates

$$(1.12) \qquad (\pi R^4)^{-1}(\tau+1) \leq tP(t)^{-1} \leq (\pi r^4)^{-1}(\tau+1) \quad .$$

Thus, $tP(t)^{-1}$ as function of τ is enclosed between two straight lines both of positive slope and having a positive intercept on the vertical axis. In particular, we see that $tP(t)^{-1}$ is strictly increasing with τ and that the equality sign in (1.8) is excluded. The slope a of the curve does not increase since the curve is convex from above; but we have now shown that even the limit of this slope for $\tau \to \infty$ must still be positive.

1.4. Using theorems on the continuity of the Green's function in dependence on its domain of definition, we can prove that P(t) is differentiable in t. From the fact that $tP(t)^{-1}$ is an increasing convex function of τ, however, it follows directly and in a much more elementary way that it has a derivative everywhere except for a null set in τ. Wherever this derivative exists, it is given by (1.8). From this formula, we can easily compute the derivative of P(t) with respect to t; using (1.4), we obtain

$$(1.13) \qquad \frac{dP(t)}{dt} = \frac{1}{t} \iint\limits_{D(t)} [3f_1^2 + f_2^2]dx\,dy \quad ,$$

where f(x,y) is the stress function of D(t). Thus, P(t) is a strictly increasing function of t.

We can combine formulas (1.4) and (1.8) in order to derive the inequality

$$(1.14) \qquad \frac{d}{d\tau}[tP(t)^{-1}] \leq \tau^{-3/2}P(t)^{-1} = \frac{1}{\tau}[tP(t)^{-1}] \quad .$$

The right side term represents the slope of the straight line that joins the point considered on the curve to the origin. This fact could have been

deduced immediately from the convexity of the curve without the use of an explicit expression for the derivative of $tP(t)^{-1}$.

1.5 The torsional rigidity P has been expressed by (1.5) as the maximum value of a certain functional in a specified function class F(x,y). This fact leads to easy lower bounds for P and was utilized systematically in the preceding sections in order to obtain convexity results for the torsional rigidity. We wish to formulate now a minimum problem connected with the torsional rigidity which will provide upper bounds for P and yield complementary convexity theorems for this functional.

Let D be a given simply-connected bounded domain in the (x,y)-plane and $q_i(x,y)$ ($i = 1,2$) be a continuously differentiable vector field in it such that its divergence satisfies

$$\frac{\partial}{\partial x} q_1 + \frac{\partial}{\partial y} q_2 = -2 \qquad (1.15)$$

Let f(x,y) be the stress function of D defined in Section 1.1. We clearly have by (1.4), (1.15) and the vanishing of f on C:

$$\iint_D (f_x q_1 + f_y q_2)\,dx\,dy = 2\iint_D f\,dx\,dy = P \qquad (1.16)$$

Hence

$$\iint_D (q_1^2 + q_2^2)\,dx\,dy = -\iint_D (f_x^2 + f_y^2)\,dx\,dy + \iint_D [(f_x - q_1)^2 + (f_y - q_2)^2]\,dx\,dy + 2P \qquad (1.17)$$

and, finally, in view of (1.4)

$$\iint_D (q_1^2 + q_2^2)\,dx\,dy \geqq P \qquad (1.18)$$

where equality can hold only for $\underline{q} = \nabla f$. Thus, the torsional rigidity appears as a minimum for a functional defined for all continuously differentiable vector fields in D with the constant divergence -2. No boundary conditions are imposed upon the vector field. The extremum problems (1.5) and (1.18) stand to each other in the same relation as do Dirichlet's and Thomson's principles in potential theory. [Cf. Section 4.4.]

Let now $D(t)$ be again the family of domains obtained from $D = D(1)$ by one-sided stretching. If $f(x,y)$ is the stress function for a domain $D(t_o)$ it defines for this domain the extremum vector field $\underline{q} = \nabla f$. We can transplant this vector field into $D(t)$ and utilize it as a test vector field in the minimum problem (1.18) for this new domain. We have to transform, however, the vector field in a proper way in order to safeguard the side condition (1.15). Let us define, therefore, in $D(t)$ the vector field

(1.19) $$\frac{t}{t_o} q_1 \left(\frac{t_o}{t} x, y\right) \quad , \quad q_2\left(\frac{t_o}{t} x, y\right) \quad .$$

In view of (1.18), we then obtain the estimate

(1.20) $$P(t) \leq \iint\limits_{D(t)} \left[\left(\frac{t}{t_o}\right)^2 q_1\left(\frac{t_o}{t} x,y\right)^2 + q_2\left(\frac{t_o}{t} x,y\right)^2\right] dx\,dy$$

and, referring back to the domain $D(t_o)$, we find in the usual way:

(1.21) $$\frac{P(t)}{t} \leq \frac{t^3}{t_o^3} \iint\limits_{D(t_o)} f_x^2\, dx\,dy + \frac{1}{t_o} \iint\limits_{D(t_o)} f_y^2\, dx\,dy \quad .$$

This inequality leads to

Theorem 1a: Under the assumptions of Theorem 1 the functional $t^{-1}P(t)$ is an increasing function of the variable $s = t^2$ and is convex from above.

The complementary character of Theorems 1 and 1a is seen best if we express the convexity statements as differential inequalities for $P(t)$. We may express Theorem 1 in the form

(1.22) $$P'' + 5\frac{P'}{t} \geq 2\frac{P'^2}{P} + 3\frac{P}{t^2}$$

while Theorem 1a leads to

(1.23) $$P'' + \frac{3}{t^2} P \leq 3\frac{P'}{t} \quad .$$

We see that the theorems provide upper and lower estimates for $P''(t)$, respectively.

The method applied in this section may be called the transplantation of an extremal vector field. It is of use in the case of a Thomson principle for a given functional while the method of transplanting an extremal (scalar) function is appropriate in the case of a Dirichlet principle. Friedrichs [5] has studied Legendre type transformations which can bring a wide class of minimum problems with prescribed boundary conditions into an equivalent form involving a maximum problem with natural boundary conditions. By such transformations one can frequently obtain for the same functional a Thomson as well as a Dirichlet type extremum characterization. The duality between the two extremum problems is reflected in the duality between the two transplantation methods.

In transplanting a vector field we have to change the components of the field as well as the independent variables of the field as is apparent in (1.19). The transplantation of such a field under general linear transformations of the independent variables will be discussed in Chapter IV. Various further applications of the method will be given in the following sections of this chapter.

2. Virtual Mass.

2.1. Let B be a finite body in the (x,y,z)-space bounded by a smooth surface S. Suppose that B is immersed in an incompressible fluid which flows irrotationally past the body under the influence of a dipole at infinity of strength 1 and directed along the x-axis. The velocity potential of such a flow has the form

(2.1) $$\Phi(x,y,z) = x + \varphi(x,y,z)$$

where $\varphi(x,y,z)$ is regular harmonic outside B. We have on S the boundary condition

$$(2.2) \qquad \frac{\partial \Phi}{\partial n} = 0 \quad , \quad \frac{\partial \varphi}{\partial n} = -\cos(n,x)$$

which determines $\varphi(x,y,z)$ in a unique way.

Let D be the exterior of B; we define [15,17]

$$(2.3) \qquad W_x = \iiint_D (\nabla\varphi)^2 \, dx\, dy\, dz$$

as the virtual mass of B in the x-direction. W_x is a functional of B which plays a role in many hydrodynamical applications.

Consider a function $f(x,y,z)$ which vanishes at infinity, is continuously differentiable in D and has a finite Dirichlet integral there. Clearly

$$(2.4) \qquad \iiint_D \nabla f \cdot \nabla\varphi \, dx\, dy\, dz = -\iint_S f \frac{\partial\varphi}{\partial n}\, d\sigma = \iint_S f \cos(n,x)\, d\sigma \quad .$$

Thus, by Schwarz' inequality

$$(2.5) \qquad \iiint_D (\nabla f)^2 dx\, dy\, dz \cdot W_x \geq \left(\iint_S f \cos(n,x)\, d\sigma\right)^2$$

and the equality sign can hold only for $\nabla f \equiv c\nabla\varphi$, where c is any constant. We have, therefore, the following definition of the virtual mass by means of an extremum problem:

$$(2.6) \qquad W_x = \max \frac{\left(\iint_S f \cos(n,x)\, d\sigma\right)^2}{\iiint_D (\nabla f)^2 dx\, dy\, dz} \quad .$$

2.2 If we introduce the transformation $x' = tx$, $y' = y$, $z' = z$, we obtain again a family of bodies $B(t)$ with exteriors $D(t)$, boundary surfaces $S(t)$, and virtual masses in the x-direction $W_x(t)$. In order to study the dependence of $W_x(t)$ on t, we use once more the extremum definition (2.6) of the virtual mass and the transplantation of the correct extremum function of a domain $D(t_0)$ into a domain $D(t)$ as a competing function.

Let $\varphi(x,y,z)$ be the correct extremal function for some fixed $D(t_o)$ and consider $f(x,y,z) = \varphi(t_o t^{-1}x,y,z)$ in the domain $D(t)$. Clearly, this function has a finite Dirichlet integral and may serve to estimate $W_x(t)$ by means of the inequality (2.6). We have

$$(2.7) \qquad W_x(t) \geq \frac{\left[\iint\limits_{S(t)} \varphi(t_o t^{-1}x,y,z)\cos(n,x)d\sigma\right]^2}{\iiint\limits_{D(t)} [t_o^2 t^{-2}\varphi_1^2 + \varphi_2^2 + \varphi_3^2]dx\,dy\,dz} \qquad .$$

We change our variables in order to use $D(t_o)$ as domain of integration instead of $D(t)$; this means simply a replacement of $t_o t^{-1}x$ by x. Observe now that $\cos(n,x)d\sigma = dy\,dz$ is not affected by this change of variable. Hence, we find

$$(2.8) \qquad [tW_x(t)]^{-1} \leq \frac{\tau t_o^2 \iiint\limits_{D(t_o)} \varphi_1^2 dx\,dy\,dz + \iiint\limits_{D(t_o)} [\varphi_2^2 + \varphi_3^2]dx\,dy\,dz}{t_o\left(\iint\limits_{S(t_o)} \varphi\cos(n,x)d\sigma\right)^2} \quad ,$$

using again the variable $\tau = t^{-2}$. Thus, we have an estimate

$$(2.9) \qquad [tW_x(t)]^{-1} \leq a\tau + b$$

and, for $\tau = \tau_o = t_o^{-2}$, we have by (2.6)

$$(2.9') \qquad [t_o W_x(t_o)]^{-1} = a\,\tau_o + b \qquad .$$

This result shows that the curve $[tW_x(t)]^{-1}$ with $\tau = t^{-2}$ as independent variable has at every point a supporting line and is convex from above. Thus, it has a tangent everywhere, except for a null set; wherever the tangent exists, its slope is given by

$$(2.10) \qquad a = \frac{t}{W_x(t)^2} \iiint\limits_{D(t)} \varphi_1^2 dx\,dy\,dz \geq 0 \qquad .$$

This follows from (2.8) and the equation

$$(2.11) \qquad \iint\limits_{S(t_o)} \varphi\cos(n,x)\,d\sigma = W_x$$

which is, in turn, an immediate consequence of (2.2) and (2.3).

We conclude from (2.10) that $tW_x(t)$ is a non-decreasing function of t. Since the volume V(t) of the body B(t) is given by $V(t) = tV(1)$ we may also say that the combination VW_x is non-decreasing in t.

We summarize our results in

<u>Theorem</u> 2. Let $W_x(t)$ be the virtual mass in the x-direction of a body B(t) which arises from B(1) by one-sided stretching in the x-direction with the parameter t. Then $[tW_x(t)]^{-1}$ is, considered as a function of the variable $\tau = t^{-2}$, an increasing function and convex from above.

<u>2.3</u>. The curve $[tW_x(t)]^{-1}$ with abscissa τ behaves asymptotically quite differently from the curve $tP(t)^{-1}$ considered in the preceding paragraph. In fact, it is easily seen from (2.10) that the slope of the curve converges to zero if $\tau \rightarrow \infty$, that is, $t \rightarrow 0$. This is due to the fact that even a plane plate has a finite virtual mass if it stands under right angles to the flow direction; and for $t \rightarrow 0$ the function $W_x(t)$ will converge to the virtual mass of the plane plate obtained by projecting the body B(1) into the (y,z)-plane.

The plane plates, perpendicular to the x-direction, give an elementary family of bodies B(t) for which the explicit dependence on τ of $[tW_x(t)]^{-1}$ can actually be computed. Since $B(t) \equiv B(1)$, we see that in this particular case

(2.12) $$[tW_x(t)]^{-1} = k\tau^{1/2} \quad , \quad k \text{ independent of } \tau,$$

which is a monotonic curve convex from above.

<u>2.4</u>. We can derive further results for the virtual mass by using the method of transplanting an extremal vector field. For this purpose, we have to characterize W_x by a Thomson type extremum principle. We define in the domain D a solenoidal vector field $\underline{q} = q_1, q_2, q_3$, that is, we require the differential condition

(2.13) $\nabla q = 0$

We suppose further that $|q| = O(\frac{1}{r^2})$ as $r \to \infty$ and that

(2.14) $$\iiint_D q^2 \, dx\, dy\, dz < \infty$$

and that the boundary condition

(2.15) $q \cdot n = -\cos(n,x)$

be satisfied on S.

The velocity potential $\varphi(x,y,z)$ defined in (2.1) leads to the vector field $\nabla\varphi$ of the required type. If q is another admissible field, we have

(2.16) $$\iiint_D q^2 \, dx\, dy\, dz = \iiint_D (\nabla\varphi)^2 dx\, dy\, dz + 2\iiint_D (q - \nabla\varphi)\nabla\varphi \, dx\, dy\, dz$$
$$+ \iiint_D (\nabla\varphi - q)^2 dx\, dy\, dz \quad .$$

Using the harmonic character of φ, the conditions (2.2), (2.13), and (2.15), we find

(2.17) $$\iiint_D (q - \nabla\varphi)\cdot\nabla\varphi \, dx\, dy\, dz = 0 \quad ;$$

thus (2.16) leads to the inequality

(2.18) $$\iiint_D q^2 dx\, dy\, dz \geqslant \iiint_D (\nabla\varphi)^2 dx\, dy\, dz$$

and equality can hold only for $q = \nabla\varphi$. By the definition (2.3) of the virtual mass, we have therefore

(2.19) $$W_x = \min \iiint_D q^2 dx\, dy\, dz$$

where all solenoidal vector fields with the boundary condition (2.15) and the proper behavior at infinity are admitted. This minimum condition characterizes the velocity potential Φ and yields at the same time estimates for the virtual mass W_x; it is called the Thomson principle for the virtual mass problem. It leads to easy upper bounds for W_x, in contradistinction to the extremum problem (2,6) which permits estimates for W_x from below.

2.5 We consider now the one-sided stretching of the body $B = B(1)$ in the x-direction. Let φ be the correct velocity potential in the exterior $D(t_o)$ of the body $B(t_o)$. We transplant the vector field $\underline{q} \equiv \nabla\varphi$ into the domain $D(t)$ by putting

$$(2.20) \qquad \underline{q}_t \equiv \left(\frac{t}{t_o} q_1(\frac{t_o}{t} x,y,z) \ , \ q_2(\frac{t_o}{t} x,y,z) \ , \ q_3(\frac{t_o}{t} x,y,z) \right) \ .$$

Clearly, $\underline{q}_t$ is well-defined in $D(t)$ and solenoidal there. Its norm is integrable over $D(t)$ and it is easily verified that on the boundary $S(t)$ we have

$$(2.21) \qquad \underline{q}_t \cdot \underline{n} = - \frac{t}{t_o} \cos (n,x) \ .$$

Thus, we have to use the vector field $\frac{t_o}{t} \underline{q}_t$ as test field in (2.19). We find in the usual way

$$(2.22) \qquad \frac{W_x(t)}{t} \leq \frac{1}{t_o} \iiint_{D(t_o)} q_1^2 \, dx\,dy\,dz + \frac{t_o}{t^2} \iiint_{D(t_o)} (q_2^2 + q_3^2) dx\,dy\,dz \ .$$

This inequality shows that $t^{-1}W_x(t)$ is a convex (from above) function of $\tau = t^{-2}$; the slope of the supporting line at every point t is given by

$$(2.23) \qquad a = t \iiint_{D(t)} (q_2^2 + q_3^2) dx\,dy\,dz = t \iiint_{D(t)} [\varphi_y{}^2 + \varphi_z{}^2] dx\,dy\,dz \ .$$

We proved thus

<u>Theorem</u> 2a. Under the assumptions of Theorem 2, the functional $t^{-1}W_x(t)$ is a non-decreasing function of $\tau = t^{-2}$ which is convex from above.

We may formulate Theorems 2 and 2a in the form of the differential inequalities

$$(2.24) \qquad W_x'' + \frac{1}{t} W_x' + \frac{1}{t^2} W_x \geq 2 \frac{W_x'^2}{W_x}$$

and

$$(2.25) \qquad W_x'' + \frac{1}{t} W_x' \leq \frac{1}{t^2} W_x \ .$$

We see again that the two results lead to upper and lower estimates for W''_x. We can combine both inequalities and obtain

(2.26) $$(W'_x/W_x)^2 \leq 1/t^2 \quad .$$

3. The Outer Radius.

3.1 Let C be a smooth closed curve in the complex z-plane; let D be the exterior of C There exists a unique univalent analytic function in D which maps D onto the exterior of the unit circle and which has, near infinity, the series development

(3.1) $$f(z) = \frac{1}{r} z + a_0 + \frac{a_1}{z} + \ldots \quad , \quad r > 0 \quad .$$

The coefficient $r(C) = r$, a functional of the curve C (or its exterior D), is called the outer radius of C.

Since

(3.2) $$g(z) = \log |f(z)| = \log |z| + \log \frac{1}{r} + 0(\frac{1}{|z|})$$

is the Green's function of D with the logarithmic pole at infinity, we may define also the outer radius potential-theoretically by

(3.2′) $$\log \frac{1}{r} = \lim_{|z| \to \infty} \left(g(z) - \log |z|\right) \quad .$$

If $z = z(s)$ is the parametric representation of the curve C in terms of its length parameter s, we define for $z(s) \in C$

(3.3) $$\nu(s) = \frac{1}{2\pi} \frac{\partial g(z)}{\partial n}$$

where n is the interior normal with respect to D. We clearly have

(3.4) $$\int_C \nu(s) ds = 1 \quad .$$

It is also easy to derive from (3.3) and Green's identity the formula

(3.5) $$\int_C \log \frac{1}{|z - \zeta|} \nu(s) ds = \log \frac{1}{r} - g(\zeta) \quad .$$

This result holds still for $\zeta \in C$ and we find

$$(3.6) \qquad \log \frac{1}{r} = \int_C \int_C \log \frac{1}{|z_1 - z_2|} \nu(s_1)\nu(s_2) ds_1 ds_2$$

with $z_1 = z(s_1)$, $z_2 = z(s_2)$.

Let next $\rho(s)$ be an arbitrary continuous function with

$$(3.7) \qquad \int_C \rho(s) ds = 0 \quad .$$

Consider the integral

$$(3.8) \qquad \mathcal{Y}(\xi, \eta) = \int_C \log \frac{1}{|z - \zeta|} \rho(s) ds \quad , \quad \zeta = \xi + i\eta \quad ,$$

which represents a harmonic function $h_e(\xi, \eta)$ in D and another harmonic function $h_i(\xi, \eta)$ in the interior $\tilde{D}$ of C. The two harmonic functions have the same limit values at each point $\zeta \in C$, but their normal derivatives jump by the amount $2\pi \rho(s)$ if we cross the curve C at the point $z(s)$.

By Green's identity, we have the relation

$$(3.9) \qquad \iint_D (\nabla h_e)^2 d\xi d\eta + \iint_{\tilde{D}} (\nabla h_i)^2 d\xi d\eta = -\int_C h_e \left[\frac{\partial h_e}{\partial n} - \frac{\partial h_i}{\partial n}\right] ds$$

$$= 2\pi \int_C \int_C \log \frac{1}{|z_1 - z_2|} \rho(s_1) \rho(s_2) ds_1 ds_2 \geqslant 0 \quad .$$

Finally, we observe that by (3.5) and (3.7)

$$(3.10) \qquad \int_C \int_C \log \frac{1}{|z_1 - z_2|} \nu(s_1) \cdot \rho(s_2) ds_1 ds_2 = 0 \quad .$$

Let now $\mu(s)$ be continuous on C and satisfy

$$(3.11) \qquad \int_C \mu \, ds = 1 \quad .$$

Then, using $\rho(s) = \nu(s) - \mu(s)$, we easily see that in view of (3.6), (3.9), and (3.10):

$$(3.12) \qquad \log \frac{1}{r} \leq \int_C \int_C \log \frac{1}{|z_1 - z_2|} \mu(s_1) \mu(s_2) ds_1 ds_2 \quad .$$

Thus, the outer radius can be defined by an extremum problem with respect to continuous functions on C with integral 1.

The extremum problem defining $\log \frac{1}{r}$, already considered by Gauss, has a simple electrostatic interpretation. In fact, consider a charge distribution $\mu(s)$ with logarithmic potential, total amount 1 and spread over the curve C. The integral in (3.12) represents the total energy of this charge distribution. The equilibrium charge distribution $\nu(s)$ is distinguished by the fact that it minimizes this energy. The minimum energy $\log \frac{1}{r}$ may be called the capacity of the conductor C.

3.2 We consider now a family of curves C(t) which are obtained from one fixed curve C(1) by one-sided stretching with parameter t. Let $C(t_o)$ be one fixed curve and $\nu(s)$ its corresponding charge density. If C(t) is any other curve of the family we define on it a density by

$$\mu(s')ds' = \nu(s)ds \qquad , \tag{3.13}$$

where s′ is the corresponding length parameter on C(t). In view of (3.12), we have

$$\log \frac{1}{r(t)^2} \leq \int_{C(t)} \int_{C(t)} \log \frac{1}{(x_1' - x_2')^2 + (y_1' - y_2')^2} \, \mu(s_1')\, \mu(s_2')ds_1'ds_2' \quad . \tag{3.14}$$

Referring back to the curve $C(t_o)$, we find easily

$$\log \frac{t^2}{r(t)^2} \leq \int_{C(t_o)} \int_{C(t_o)} \log \frac{t^2}{\frac{t^2}{t_o^2}(x_1 - x_2)^2 + (y_1 - y_2)^2} \, \nu(s_1)\nu(s_2)ds_1ds_2 \ . \tag{3.15}$$

We may interpret

$$\exp\left[\int_{C(t_o)} \int_{C(t_o)} \log \frac{t^2\, t_o^2}{t^2(x_1 - x_2)^2 + t_o^2(y_1 - y_2)^2} \, \nu(s_1)\, \nu(s_2)ds_1ds_2 \right] \tag{3.16}$$

$$= G\left[\frac{t^2\, t_o^2}{t^2(x_1 - x_2)^2 + t_o^2(y_1 - y_2)^2}\right]$$

as a geometric mean with the non-negative weight function $\nu(s)$; thus, we can write:

$$\frac{t^2}{r(t)^2} \leq G\left[\frac{t^2\, t_o^2}{t^2(x_1 - x_2)^2 + t_o^2(y_1 - y_2)^2}\right] \tag{3.17}$$

We apply this result for two different values of t, say for t and T. Using the well-known Hölder-Minkowski inequality [7, p. 21, (2.7.1)]

$$G[a] + G[b] \leq G[a + b] \tag{3.18}$$

we find

$$\frac{t^2}{r(t)^2} + \frac{T^2}{r(T)^2} \leq G\left[\frac{t^2\, t_o^2}{t^2(x_1 - x_2)^2 + t_o^2(y_1 - y_2)^2} + \frac{T^2\, t_o^2}{T^2(x_1 - x_2)^2 + t_o^2(y_1 - y_2)^2}\right] . \tag{3.19}$$

Now, we utilize the fact that if $A > 0$, $B > 0$, and $w > 0$

$$\frac{d^2}{dw^2}\left[\frac{w}{Aw + B}\right] = - \frac{2AB}{(Aw + B)^3} < 0 \quad . \tag{3.20}$$

This shows that $w/(Aw+B)$ is convex from above and satisfies the subadditivity condition

$$\frac{1}{2}\left[\frac{w_1}{Aw_1 + B} + \frac{w_2}{Aw_2 + B}\right] \leq \frac{\frac{1}{2}(w_1 + w_2)}{A\cdot\frac{1}{2}(w_1 + w_2) + B} \quad . \tag{3.21}$$

Let us suppose that t, t_o, and T are chosen so that

$$t_o^2 = \frac{1}{2}(t^2 + T^2) \quad . \tag{3.22}$$

Then we find from (3.19) and (3.21):

$$\frac{1}{2}\left[\frac{t^2}{r(t)^2} + \frac{T^2}{r(T)^2}\right] \leq G\left[\frac{t_o^2}{(x_1 - x_2)^2 + (y_1 - y_2)^2}\right] \quad . \tag{3.23}$$

But $\nu(s)$ was just the equilibrium charge distribution of the curve $C(t_o)$ and hence by (3.6) the right side of (3.23) is just $t_o^2/(r(t_o)^2)$. Thus, we finally proved:

$$\frac{1}{2}\left[\frac{t^2}{r(t)^2} + \frac{T^2}{r(T)^2}\right] \leq \frac{\frac{1}{2}(t^2 + T^2)}{r(\frac{1}{2}(t^2 + T^2))} \tag{3.24}$$

that is, $t^2/(r(t)^2)$ is convex from above as function of t^2.

We can transform this result by observing the identity

$$\frac{d^2}{d\zeta^2} f(x) = x^3 \frac{d^2}{dx^2} [xf(x)] \quad , \quad \zeta = 1/x \tag{3.25}$$

which holds for every twice differentiable function $f(x)$. This formula shows that from the convexity of $xf(x)$ in $x > 0$ follows the convexity of $f(x)$ in dependence of $\zeta = 1/x > 0$. Thus, (3.24) leads to the result that $r(t)^{-2}$ is convex from above as function of $\tau = t^{-2}$.

Finally, it is easily seen from (3.14) that $r(t)^{-2}$ does not decrease with increasing τ. Thus, we may state

<u>Theorem</u> 3. Let $C(t)$ be the smooth closed plane curve which arises from $C(1)$ by one-sided stretching in a fixed direction with the parameter t; let $r(t)$ be its outer radius. Then $r(t)^{-2}$ is non-decreasing and convex from above as a function of the variable $\tau = t^{-2}$.

<u>3.3</u>. It is easy to extend Theorem 3 to arbitrary continua in the plane by approximating them by smooth closed curves for which the result has been established. There exist two continua for which $r(t)^{-2}$ is a linear function of τ. In fact, consider any line segment of length ℓ; it is easily seen that its outer radius is $r = \ell/4$. If the segment lies in the direction of the stretching, we have

$$r(t)^{-2} = \frac{16}{\ell^2} \tau \quad , \tag{3.26}$$

while if it lies in normal direction $r(t)^{-2}$ will be unaffected by the stretching. Thus, in the inequality

$$\frac{d^2}{d\tau^2} r(t)^{-2} \leq 0 \tag{3.27}$$

equality does hold for these two types of segments.

If $C(1)$ is a circle of radius R the curves $C(t)$ will be ellipses with semi-axes R and Rt. It is easy to compute

$$r(t)^{-2} = \frac{4}{R^2}(1+t)^{-2} = \frac{4}{R^2}\frac{\tau}{(\sqrt{\tau}+1)^2} \qquad (3.28)$$

This formula enables us to determine the asymptotic behavior of $r(t)^{-2}$ in the case that $\tau \to 0$ or ∞ for many curves $C(t)$. In fact, let us suppose that $C(1)$ contains a circle of radius R_o in its interior and is enclosed by another circle R_1. The outer radius r is a monotonic set function as can be seen easily from (3.2) and the maximum principle. Thus, by means of (3.28), we have the estimates

$$\frac{4}{R_1^2}(1+t)^{-2} \leq r(t)^{-2} \leq \frac{4}{R_o^2}(1+t)^{-2} \qquad (3.29)$$

3.4. We defined the capacity log 1/r of the curve C as the minimum energy of a charge distribution of total amount 1 spread over C. Suppose now that we bring n charged particles on the conductor C, each having the charge 1/n; what is the minimum energy of this set? Clearly, we have to solve the minimum problem:

$$\frac{1}{2n^2}\sum_{i \neq k} \log \frac{1}{|z_i - z_k|} = \min \quad , \qquad i,k = 1,\ldots,n \qquad (3.30)$$

if $z_i \in C$ is the location of the i^{th} particle in the equilibrium position.

Closely related to this problem is the question for the n^{th} order diameter of a curve C which is defined by [3]

$$[d_n(C)]^{\binom{n}{2}} = \max \prod_{i<k} |z_i - z_k| \; ; \quad i,k = 1,\ldots,n; z_i \in C,\ z_k \in C \qquad (3.31)$$

It can easily be shown that the $d_n(C)$ form a montonically decreasing sequence and that they converge to the outer radius r of C.

We can deal with the d_n in the same way as we dealt with the outer radius. Transplanting the extremum points z_i under a stretching, we can prove that d_n^{-2} is a non-decreasing convex function of $\tau = t^{-2}$.

We remark, finally, that the d_n can be defined for an arbitrary plane point set and that $\lim_{n\to\infty} d_n$ does always exist. This limit is called the transfinite diameter d of the set; [3]. It can be shown from our preceding result that d^{-2} is a non-decreasing function of $\tau = t^{-2}$ and convex from above.

4. Electrostatic Capacity.

4.1. Let S be a smooth closed surface in the (x,y,z)-space and let D be its exterior. We define in D a harmonic function V(x,y,z) which satisfies on S the boundary condition

$$(4.1)\qquad V = 1 \quad \text{on } S.$$

V is called the conductor potential of the surface S; it has at infinity the form

$$(4.1')\qquad V(x,y,z) = \frac{K}{r} + O\left(\frac{1}{r^2}\right) \quad , \quad r = \sqrt{x^2 + y^2 + z^2}$$

and the factor K is called the capacity of S. K may be interpreted as the electric charge which induces on the conductor S the potential 1.

By Green's theorem, we have

$$(4.2)\qquad \iiint_D (\nabla V)^2 dx\,dy\,dz = -\iint_S V \frac{\partial V}{\partial n} d\sigma = 4\pi K \quad .$$

If f(x,y,z) is an arbitrary function which is continuously differentiable in D, has a finite Dirichlet integral over D and vanishes on S and at infinity, we have

$$(4.3)\qquad \iiint_D \nabla V \cdot \nabla f\, dx\,dy\,dz = 0 \quad .$$

Thus, clearly by the usual arguments

$$(4.4)\qquad K = \min \frac{1}{4\pi} \iiint_D (\nabla U)^2 dx\,dy\,dz$$

where all continuously differentiable functions U(x,y,z) with finite Dirichlet integral are admitted which have on S the boundary value 1 and

which vanish at infinity. This minimum principle characterizes at the same time the conductor potential V and the capacity K. It is called the Dirichlet principle for the conductor problem.

If $S(1)$ is a fixed surface, we may obtain the family $S(t)$ by one-sided stretching, say in the x-direction, and denote by $K(t)$ the corresponding value of the capacity. From the definition of K by means of the extremum problem (4.4), we can derive the following results by the method of transplanting the extremal function:

$$\frac{K(t)}{t} \leq \frac{1}{4\pi t_o} \iiint_{D(t_o)} [\frac{t_o^2}{t^2} V_x^2 + V_y^2 + V_z^2] dx\, dy\, dz \tag{4.5}$$

where $V(x,y,z)$ is the conductor potential of the surface $S(t_o)$. Thus, we have

<u>Theorem</u> 4. Let $S(t)$ be a closed smooth surface which arises from $S(1)$ by one-sided stretching in a fixed direction with the parameter t; let $K(t)$ be its capacity. Then $K(t)t^{-1}$ is a non-decreasing function of $\tau = t^{-2}$ and convex from above.

The slope of the supporting line at every point t of the curve is given by

$$a = \frac{t}{4\pi} \iiint_{D(t)} V_x^2 \, dx\, dy\, dz \quad . \tag{4.6}$$

From this formula we can easily compute

$$\frac{d}{dt} K(t) = \frac{1}{4\pi t} \iiint_{D(t)} [-V_x^2 + V_y^2 + V_z^2] dx\, dy\, dz \quad . \tag{4.7}$$

<u>4.2</u>. We can define on the surface S the charge density

$$\nu(P) = -\frac{1}{4\pi K} \frac{\partial V}{\partial n} \tag{4.8}$$

which is by the maximum principle non-negative on S and satisfies the condition

$$\iint_S \nu(P) d\sigma = 1 \quad . \tag{4.9}$$

If P and Q are points of D we shall denote their distance by $r(P,Q)$. We have

(4.10) $$\frac{1}{K} V(Q) = \iint_S \nu(P) r(P,Q)^{-1} d\sigma_P$$

and by virtue of (4.9):

(4.11) $$\frac{1}{K} = \iint_S \iint_S \nu(P)\nu(Q) r(P,Q)^{-1} d\sigma_P d\sigma_Q \quad .$$

It is also easy to verify Gauss' theorem:

(4.12) $$\frac{1}{K} = \min \iint_S \iint_S \mu(P)\mu(Q) r(P,Q)^{-1} d\sigma_P d\sigma_Q$$

where all functions $\mu(P)$ on S are admitted which satisfy

(4.13) $$\iint_S \mu(P) d\sigma = 1 \quad .$$

We obtain from the extremum definition (4.12) for the capacity an estimate for the quantity $K(t)$. Transplanting the correct charge distribution $\nu(P)$ of $S(t_o)$ onto $S(t)$ by the law

(4.14) $$\mu(P) d\sigma' = \nu(P) d\sigma \quad ,$$

where $d\sigma'$ and $d\sigma$ are corresponding surface elements, we obtain:

(4.15) $$\frac{1}{K(t)} \leq \iint_{S(t_o)} \iint_{S(t_o)} \left[\frac{t^2}{t_o^2}(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\right]^{-1/2} \nu(P)\nu(Q) d\sigma_P d\sigma_Q .$$

We remark next that if a, b, and t are positive

(4.16) $$\frac{d^2}{dt^2}\left(\frac{t}{\sqrt{at^2+b}}\right) < 0 \quad .$$

Hence, we obtain from (4.15):

(4.17) $$\frac{t}{K(t)} \leq \frac{t_o}{K(t_o)} + (t-t_o) \iint_{S(t_o)} \iint_{S(t_o)} \frac{(y-\eta)^2+(z-\zeta)^2}{r(P,Q)^3} \nu(P)\nu(Q) d\sigma_P d\sigma_Q \quad .$$

This estimate proves that $tK(t)^{-1}$ is a non-decreasing function of t which is convex from above. This can also be formulated as the statement that $K(t)^{-1}$ is convex from above in dependence on t^{-1}. (See (3.25).) If S

is a plane surface perpendicular to the x-direction, we have $S(t) \equiv S(1)$ and in this case $tK(t)^{-1}$ becomes a linear function of t while $K(t)^{-1}$ is independent of t^{-1}. Thus our result is, in a certain sense, a sharp one.

By (4.17), we find as the slope of the supporting line to the curve $tK(t)^{-1}$ versus t

$$(4.18) \qquad a = \iint_{S(t_o)} \iint_{S(t_o)} \frac{(y-\eta)^2 + (z-\zeta)^2}{r(P,Q)^3} \nu(P)\nu(Q) d\sigma_P d\sigma_Q \quad .$$

We can compute from this result

$$(4.19) \qquad \frac{d}{dt} K(t) = \frac{K(t)^2}{t} \iint_{S(t_o)} \iint_{S(t_o)} \frac{\nu(P)\nu(Q)}{r(P,Q)^3} (x-\xi)^2 d\sigma_P d\sigma_Q \quad .$$

This formula shows that the capacity grows with a one-sided stretching.

Comparing formulas (4.7) with (4.19) we obtain an interesting integral identity. It leads, in particular, to the inequality

$$(4.20) \qquad \iiint_D V_x^2 \, dx\,dy\,dz \leq \iiint_D [V_y^2 + V_z^2] dx\,dy\,dz \quad .$$

Equality holds here if the surface S lies entirely in the (y,z)-plane, since in this case we have obviously $K'(t) = 0$.

In order to compare the convexity results of the last two sections, we formulate them as differential inequalities for K(t). The convexity of $K(t)t^{-1}$ in dependence on t^{-2} leads to the inequality:

$$(4.21) \qquad t^2 \frac{d^2K}{dt^2} + t \frac{dK}{dt} \leq K$$

while the convexity of $tK(t)^{-1}$ in dependence on t yields:

$$(4.22) \qquad (t \frac{d^2K}{dt^2} + 2 \frac{dK}{dt})K \geq 2tK'^2 \quad .$$

We see that the two inequalities lead to estimates of $\frac{d^2K}{dt^2}$ from both sides and thus complement each other. This was to be expected since they were derived from the complementary extremum problems (4.4) and (4.12) which

permit upper and lower estimates for the capacity.

4.3. The extremum problem (4.4) for the capacity K can be generalized by admitting continuous functions $\mathcal{U}(x,y,z)$ in D which are continuously differentiable except for a smooth surface $\sum$ in D across which their derivatives are permitted to jump. In fact, the equation (4.3) holds for functions f(x,y,z) of this class and this implies (4.4).

The fact that we can relax the continuity requirements for the competing functions allows us to introduce a type of one-sided stretching slightly more general than that used till now. We consider the transformation of the (x,y,z)-space

$$x' = \begin{cases} tx & \text{for } x \geq 0 \\ x & \text{for } x \leq 0 \end{cases}, \quad y' = y, \quad z' = z \tag{4.3}$$

which is one-to-one and continuous but which has discontinuous derivatives across the plane $x = 0$ for $t \neq 1$. We call this transformation a partial one-sided stretching in the x-direction with parameter t.

Let $B = B(1)$ be a given body with surface $S = S(1)$, exterior $D = D(1)$, and capacity $K = K(1)$. By $B(t)$, $S(t)$, $D(t)$, and $K(t)$, we denote the corresponding concepts after a partial one-sided stretching in the x-direction with parameter t. Let $V(x,y,z)$ denote the correct conductor potential for the body $B(t_o)$; then the function

$$V_t = \begin{cases} V(\frac{t_o}{t} x,y,z) & \text{for } x \geq 0 \\ V(x,y,z) & \text{for } x \leq 0 \end{cases} \tag{4.24}$$

will be well-defined in $D(t)$ and an admissible test function for the minimum problem (4.4). Hence

$$(4.25)\quad 4\pi K(t) \leq \iiint\limits_{D(t),x\geq 0} [(\frac{t_o}{t})^2 \; V_x^2(\frac{t_o}{t}\, x,y,z) + V_y^2 + V_z^2]\,dx\,dy\,dz$$

$$+ \iiint\limits_{D(t),x\leq 0} [V_x^2 + V_y^2 + V_z^2]\,dx\,dy\,dz \quad .$$

We refer back to $D(t_o)$ and denote the part of $D(t_o)$ in the half space $x \geq 0$ by D_o^+ and the other part by D_o^-. We find

$$(4.26)\quad 4\pi K(t) \leq \iiint\limits_{D_o^-} [V_x^2 + V_y^2 + V_z^2]\,dx\,dy\,dz + \iiint\limits_{D_o^+} [\frac{t_o}{t}\, V_x^2 + \frac{t}{t_o}\,(V_y^2 + V_z^2)]\,dx\,dy\,dz$$

with $V = V(x,y,z)$. We rearrange the terms and observe that

$$(4.27)\qquad 4\pi K(t_o) = \iiint\limits_{D(t_o)} [V_x^2 + V_y^2 + V_z^2]\,dx\,dy\,dz \quad ;$$

thus, we obtain

$$(4.28)\qquad K(t) - K(t_o) \leq \frac{t - t_o}{4\pi t_o} \iiint\limits_{D_o^+} [V_y^2 + V_z^2 - V_x^2]\,dx\,dy\,dz$$

$$+ \frac{(t - t_o)^2}{4\pi t\, t_o} \iiint\limits_{D_o^+} V_x^2 \, dx\,dy\,dz \quad .$$

We use the well-known fact that $K(t)$ has arbitrarily many derivatives with respect to t. Hence, (4.28) implies immediately

$$(4.29)\qquad K'(t_o) = \frac{1}{4\pi t_o} \iiint\limits_{D_o^+} [V_y^2 + V_z^2 - V_x^2]\,dx\,dy\,dz \quad .$$

This formula holds even when the whole body B lies in the half-space $x < 0$; clearly, in this case, the bodies $B(t)$ will coincide with B and hence $K'(t) \equiv 0$. This leads to the integral identity

$$(4.30)\qquad \iiint\limits_{H} [V_y^2 + V_z^2 - V_x^2]\,dx\,dy\,dz = 0$$

if H is any half-space in D bounded by a plane $x = \text{const}$. This identity can also be verified directly by integration by parts.

We obtain further from (4.28) the estimate for the second derivative of $K(t)$:

(4.31) $$K''(t_o) \leq \frac{1}{2\pi t_o^2} \iiint_{D_o^+} V_x^2 \, dx\,dy\,dz \quad .$$

Finally, combining (4.29) and (4.31), we find

(4.32) $$t_o^2 K''(t_o) + t_o K'(t_o) \leq \frac{1}{4\pi} \iiint_{D_o^+} [V_x^2 + V_y^2 + V_z^2] dx\,dy\,dz \leq K(t_o) \quad .$$

Thus, we arrive at the differential inequality

(4.32') $$t^2 K''(t) + tK'(t) \leq K(t) \quad ,$$

which generalizes (4.21) to the case of partial stretching. We find thus:

<u>Theorem</u> 4a: The functional $t^{-1}K(t)$ is convex from above in dependence of the variable $\tau = t^{-2}$ in the case of a partial one-sided stretching with the parameter t.

<u>4.4</u>. We formulate now the Thomson principle for the capacity of a conductor. We observe that $\frac{1}{4\pi K} \nabla V$ represents a vector field $\underline{q_o}$ in D with the following properties:

(4.33) $$\nabla \cdot \underline{q} = 0$$

(4.34) $$\iiint_D q^2 \, dx\,dy\,dz < \infty \quad , \quad |q| = O(r^{-2}) \quad \text{as} \quad r \to \infty,$$

(4.35) $$\iint_{\rho} \underline{q} \cdot \underline{n} \, d\sigma = 1 \quad .$$

If $\underline{q}(x,y,z)$ is an arbitrary vector field with continuously differentiable components in D and satisfying the above three conditions, we have

(4.36) $$\iiint_D q^2 dx\,dy\,dz = \iiint_D \underline{q_o}^2 dx\,dy\,dz + 2\iiint_D \underline{q_o} \cdot (\underline{q} - \underline{q_o}) dx\,dy\,dz + \iiint_D (\underline{q} - \underline{q_o})^2 dx\,dy\,dz \quad .$$

We observe next that by (4.33) and (4.35) and the fact $V = 1$ on S

(4.37) $$\iiint_D q_o \cdot q \, dx\,dy\,dz = \frac{1}{4\pi} \iint_S \frac{1}{K} V(q \cdot n) d\sigma = \frac{1}{4\pi K}$$

and, in particular,

$$(4.37') \qquad \iiint\limits_D q_o^2 \,dx\,dy\,dz = \frac{1}{4\pi K} \quad .$$

Thus, (4.36) leads to

$$(4.38) \qquad \iiint\limits_D \underline{q}^2 \,dx\,dy\,dz \geqslant \frac{1}{4\pi K} \quad .$$

and $\frac{1}{4\pi K}$ can be defined as the minimum of the left-side integral for all admissible vector fields $\underline{q}$. This is the required Thomson principle.

It is important to observe that the vector field $\underline{q}$ may even be discontinuous along a smooth surface Σ in D, provided only that $(\underline{q}\cdot\underline{n})$ be the same on both sides of Σ. In fact, this latter condition is sufficient in order to establish (4.37) which is equivalent to Thomson's principle.

Consider now again the bodies B(t) obtained from B = B(1) by the partial one-sided stretching (4.23). Let $\underline{q} = \nabla V$ be the extremal vector field for the exterior $D(t_o)$ of the body $B(t_o)$. We define in D(t) the transplanted vector field

$$(4.39) \qquad \underline{q}_t \equiv \left(q_1\left(\frac{t_o}{t}\,x,y,z\right) \, , \, \frac{t_o}{t}\, q_2\left(\frac{t_o}{t}\,x,y,z\right) \, , \, \frac{t_o}{t}\, q_3\left(\frac{t_o}{t}\,x,y,z\right)\right)$$

if $x \geqslant 0$, while $\underline{q}_t = \underline{q}$ if $x \leqslant 0$. The vector field $\underline{q}_t$ is well defined in D(t); it is discontinuous across the plane x = 0 but its normal component passes continuously across. It satisfies the conditions (4.33) - (4.35) with respect to the domain D(t) and is, therefore, a test field in the inequality (4.38).

We arrive at the inequality

$$(4.40) \qquad \frac{1}{4\pi K(t)} \leq \iiint\limits_{D_o^+} \left[\frac{t}{t_o}\, q_1^2 + \frac{t_o}{t}\,(q_2^2 + q_3^2)\right] dx\,dy\,dz$$

$$+ \iiint\limits_{D_o^-} [q_1^2 + q_2^2 + q_3^2]\,dx\,dy\,dz \quad .$$

Rearranging as in the preceding section, we find

$$(4.41)\qquad \frac{1}{4\pi K(t)} - \frac{1}{4\pi K(t_o)} \leq \frac{t_o - t}{t} \iiint_{D_o^+} [q_2^2 + q_3^2 - q_1^2] dx\, dy\, dz + \frac{(t-t_o)^2}{tt_o} \iiint_{D_o^+} q_1^2\, dx\, dy\, dz \quad .$$

We deduce from (4.41)

$$(4.42)\qquad \frac{d}{dt}\left[\frac{1}{4\pi K(t)}\right] = -\frac{1}{t} \iiint_{D(t)^+} [q_2^2 + q_3^2 - q_1^2] dx\, dy\, dz$$

which agrees with (4.29). Furthermore, we derive from the same inequality

$$(4.43)\qquad \frac{d^2}{dt^2}\left[\frac{1}{4\pi K(t)}\right] \leq \frac{2}{t^2} \iiint_{D^+} q_1^2\, dx\, dy\, dz - \frac{2}{t}\frac{d}{dt}\left[\frac{1}{4\pi K(t)}\right] \quad .$$

Combining (4.42) and (4.43) we find

$$(4.44)\qquad \left[\frac{1}{4\pi K(t)}\right]'' \leq \frac{2}{t^2} \iiint_{D(t)^+} [q_2^2 + q_3^2] dx\, dy\, dz$$

and

$$(4.45)\qquad \left[\frac{1}{4\pi K(t)}\right]'' + \frac{1}{t}\left[\frac{1}{4\pi K(t)}\right]' \leq \frac{1}{t^2} \iiint_{D(t)^+} [q_1^2 + q_2^2 + q_3^2] dx\, dy\, dz \leq \frac{1}{t^2}\left[\frac{1}{4\pi K}\right] \quad .$$

This inequality may be interpreted as follows:

<u>Theorem</u> 4b: Under partial one-sided stretching the functional $tK(t)^{-1}$ is convex from above as a function of the variable $s = t^2$.

This result holds, of course, a fortiori in the case of one-sided stretching of the whole space; it is, however, a weaker result than the convexity of $tK(t)^{-1}$ as a function of t which was proved in Section 4.2.

5. Angular Stretching.

5.1. It is possible to study the change of the preceding and similar functionals for various other types of deformations in the same way as we did in the case of one-sided stretching. The extremum property of the functional combined with the transplantation of the extremal function leads again to statements on monotonic or convex dependence on the parameter of the deformation.

We mention, as an example, the deformation by angular stretching. Let D be a domain in the (x,y)-plane and let O be a point outside D. We choose O as the origin of a polar coordinate system r, φ and assume that D appears from O under an angle $\delta < 2\pi$. We subject the whole plane to the deformation

$$(5.1) \qquad r' = r \quad , \quad \varphi' = t\varphi \quad , \quad 0 < t < \frac{2\pi}{\delta}$$

and define $D(t)$ as the image of D under this transformation. Clearly, $D(t)$ lies still simply over the plane and the mapping $D \to D(t)$ is one-to-one. We shall say that $D(t)$ has been obtained from $D = D(1)$ by angular stretching with the parameter t.

The torsional rigidity $P(t)$ of the simply-connected domain $D(t)$ can be estimated by means of the inequality (1.5) and by using the transplanted stress function from the domain $D(t_o)$. It is useful to bring the formula (1.5) into polar coordinates and to consider the functions $F = F(r,\varphi)$. We have

$$(5.2) \qquad P = \max \frac{\left(2 \iint\limits_D F(r,\varphi) r \, dr \, d\varphi\right)^2}{\iint\limits_D [F_r^2 + \frac{1}{r^2} F_\varphi^2] r \, dr \, d\varphi} \quad .$$

Using the same method as in Section 1.1, we find the inequality

$$(5.3)\qquad \frac{t}{P(t)} \leq t_o \frac{\tau t_o^2 \iint\limits_{D(t_o)} \frac{1}{r^2} f_\varphi^2 \, r\,dr\,d\varphi + \iint\limits_{D(t_o)} f_r^2 \, r\,dr\,d\varphi}{\left(2 \iint\limits_{D(t_o)} f(r,\varphi)\, r\,dr\,d\varphi\right)^2}$$

where $\tau = t^{-2}$ and $f(r,\varphi)$ is the correct stress function for the domain $D(t_o)$.

We see from (5.3) that the curve of $tP(t)^{-1}$ plotted versus the abscissa $\tau = t^{-2}$ has at every point a supporting line with the slope

$$(5.4)\qquad a = t^3 P(t)^{-2} \iint\limits_{D(t)} \frac{1}{r^2} f_\varphi^2 \, r\,dr\,d\varphi > 0 \quad .$$

W may thus state

Theorem 5. Let $P(t)$ be the torsional rigidity of the simply-connected bounded domain $D(t)$ obtained from $D = D(1)$ by angular stretching with the parameter t. Then $tP(t)^{-1}$ is increasing and convex from above as a function of the variable $\tau = t^{-2}$.

5.2. We obtain from (5.4) and (1.4) the estimate

$$(5.5)\qquad \frac{d}{d\tau}[tP(t)^{-1}] < \frac{1}{\tau}[tP(t)^{-1}] \quad .$$

This inequality can be written as

$$(5.6)\qquad \frac{d}{d\tau} \log [t^3 P(t)^{-1}] < 0$$

which shows that $t^3 P(t)^{-1}$ decreases with increasing τ or that

(5.7) $\quad P(t)t^{-3}$ decreases with increasing t

while

(5.8) $\quad P(t)t^{-1}$ increases with t.

These two estimates give upper and lower bounds for $P(t)$ if $P(1)$ is known.

We can derive from (5.4) also the derivative of $P(t)$ with respect to t and obtain

$$(5.9)\qquad \frac{dP(t)}{dt} = \frac{1}{t} \iint\limits_{D(t)} [f_r^2 + \frac{3}{r^2} f_\varphi^2] r\,dr\,d\varphi \quad ,$$

a formula very similar to (1.13).

Chapter II

On a Theorem of Poincaré

1. Statement and Proof.

1.1. We consider real valued functions u, v, $u_1, u_2, \ldots$ φ, $\varphi_1, \ldots$ that belong to a certain class Σ. We assume that if u and v belong to Σ and c is a real constant, also u+v and cu belong to Σ. We consider a symmetric bilinear functional $A[u,v]$ over Σ. That is, $A[u,v]$ is defined for any functions u and v of the class Σ and has the properties

(1.1) $$A[u_1 + u_2, v] = A[u_1, v] + A[u_2, v] \quad ,$$

(1.2) $$A[cu, v] = cA[u, v] \quad ,$$

(1.3) $$A[u, v] = A[v, u] \quad .$$

Let $B[u,v]$ denote another symmetric bilinear functional over Σ. (Therefore, (1.1), (1.2), and (1.3) remain valid if we substitute B for A.) We assume that B is definite positive; that is,

(1.4) $$B[u,u] > 0$$

unless u vanishes identically. With the functionals $A[u,v]$ and $B[u,v]$ we form the Rayleigh ratio

(1.5) $$R[u] = \frac{A[u,u]}{B[u,u]} \quad ;$$

$R[u]$ is defined for all functions u belonging to Σ, except for the identically vanishing function. If c is any constant, $c \neq 0$,

(1.6) $$R[cu] = R[u] \quad ,$$

according to (1.2), (1.3).

We assume that we deal with one of the many examples, familiar in mathematical physics, in which it is possible to define the eigenvalues $\nu_1, \nu_2, \nu_3, \ldots$ and the eigenfunctions $\varphi_1, \varphi_2, \varphi_3, \ldots$ belonging to $R[u]$ as follows:

The first eigenvalue is the minimum of $R[u]$ and φ_1 is a function for which this minimum is attained, so that

(1.7) $$\nu_1 = R[\varphi_1] \leq R[u]$$

for any function u of Σ ($u \equiv 0$ excepted, of course). In view of (1.6) we can choose φ_1 so that

(1.8) $$B[\varphi_1, \varphi_1] = 1 \quad .$$

Now assume that $n \geq 2$ and $\varphi_1, \varphi_2, \ldots, \varphi_{n-1}$ are already defined so that

(1.9) $$B[\varphi_i, \varphi_k] = \begin{cases} 0 & i \neq k \\ 1 & i = k \end{cases}$$

for $i,k = 1,2,\ldots,n-1$. Let us seek the minimum of $R[u]$ under the side-conditions

(1.10) $$B[\varphi_1, u] = B[\varphi_2, u] = \ldots = B[\varphi_{n-1}, u] = 0 \quad .$$

Let ν_n be the value of this minimum and φ_n one of the functions for which ν_n is attained, so that the relations (1.10) imply

(1.11) $$\nu_n = R[\varphi_n] \leq R[u] \quad .$$

We may choose φ_n so that

(1.12) $$B[\varphi_n, \varphi_n] = 1 \quad .$$

Then, obviously, the relations (1.9) are satisfied for $i,k = 1,2,\ldots,n-1,n$.

1.2. There is a simple and efficient method to estimate the eigenvalues $\nu_1, \nu_2, \ldots$ that seems to have been first observed by Poincaré [11, p. 259 - 261].

Let $g_1, g_2, \ldots, g_m$ be any m linearly independent functions of the class Σ. and let the real variables $x_1, x_2, \ldots, x_m$ range through all systems of values except $0,0,\ldots,0$. Then

(1.13) $$R[x_1 g_1 + x_2 g_2 + \ldots + x_m g_m] = \frac{\sum\sum x_i x_k A[g_i, g_k]}{\sum\sum x_i x_k B[g_i, g_k]}$$

by virtue of (1.1), (1.2), (1.3), and (1.5). By virtue of (1.4), the denominator is a definite positive quadratic form in $x_1, x_2, \ldots, x_m$.

We assume that the quadratic forms $\sum\sum a_{ik}x_ix_k$ and $\sum\sum b_{ik}x_ix_k$, the latter of which is supposed to be definite positive, have the property that for all values of $x_1, x_2, \ldots, x_m$ (0,0,...,0, of course, excepted)

$$(1.14) \qquad R(x_1g_1 + x_2g_2 + \ldots + x_mg_m) \leq \frac{\sum\sum a_{ik}x_ix_k}{\sum\sum b_{ik}x_ix_k} \quad .$$

We let $\nu_1', \nu_2', \ldots, \nu_m'$ denote the roots of the characteristic equation

$$(1.15) \qquad \begin{vmatrix} a_{11} - \nu b_{11} & \cdots & a_{1m} - \nu b_{1m} \\ \cdot & \cdots & \cdot \\ a_{m1} - \nu b_{m1} & \cdots & a_{mm} - \nu b_{mm} \end{vmatrix} = 0 \quad ,$$

increasingly ordered so that

$$(1.16) \qquad \nu_1' \leq \nu_2' \leq \nu_3' \leq \ldots \leq \nu_m' \quad .$$

Then

$$(1.17) \qquad \nu_1 \leq \nu_1', \nu_2 \leq \nu_2', \ldots, \nu_m \leq \nu_m' \quad .$$

We follow the usual notation for quadratic forms in assuming that $a_{ik} = a_{ki}$, $b_{ik} = b_{ki}$.

In order to prove the theorem stated, we make use of a well-known theorem of Algebra [1, Chapter 13, Sect. 59, Theorem II]. We introduce m linearly independent linear homogeneous functions $y_1, y_2, \ldots, y_m$ of $x_1, x_2, \ldots, x_m$ so that

$$(1.18) \qquad \frac{\sum\sum a_{ik}x_ix_k}{\sum\sum b_{ik}x_ix_k} = \frac{\nu_1'y_1^2 + \nu_2'y_2^2 + \ldots + \nu_m'y_m^2}{y_1^2 + y_2^2 + \ldots + y_m^2} \quad .$$

We set

$$(1.19) \qquad u = x_1g_1 + x_2g_2 + \ldots + x_mg_m \quad ,$$

consider an integer n, $1 \leq n \leq m$, subject $x_1, x_2, \ldots, x_m$ to the n-1 conditions (1.10) and, in addition, to the m-n conditions

(1.20) $$y_{n+1} = y_{n+2} = \ldots = y_m = 0 \quad .$$

Thus we set up a system of

$$(n-1) + (m-n) = m-1$$

homogeneous linear equations for the m unknowns $x_1, x_2, \ldots, x_m$. This system possesses a solution different from $0, 0, \ldots, 0$ which we call $x_1, x_2, \ldots, x_m$. As these numbers do not all vanish and $g_1, g_2, \ldots, g_m$ are linearly independent, the function u, given by (1.19), does not vanish identically and, since it satisfies (1.10), satisfies also (1.11). We infer from (1.11), (1.14), (1.18), (1.20), and (1.16) that

(1.21) $$\begin{aligned} \nu_n &\leq R[u] \\ &\leq \frac{\sum\sum a_{ik} x_i x_k}{\sum\sum b_{ik} x_i x_k} \\ &= \frac{\nu_1' y_1^2 + \nu_2' y_2^2 + \ldots + \nu_n' y_n^2}{y_1^2 + y_2^2 + \ldots + y_n^2} \\ &\leq \nu_n' \quad , \end{aligned}$$

and so we have attained the desired conclusion $\nu_n \leq \nu_n'$ for $n = 1, 2, \ldots, m$.

2. Remarks and Applications.

2.1. It is important to observe that, in view of (1.13), we satisfy the condition (1.14) if we choose

(2.1) $$a_{ik} = A[g_i, g_k] \quad , \quad b_{ik} = B[g_i, g_k] \quad .$$

We may make the following deduction from the fact that φ_n provides the minimum of $R[\varphi]$ under the side conditions (1.10). Let Ψ be an arbitrary function of the class Σ which is orthogonal to $\varphi_1, \ldots, \varphi_{n-1}$. Then, we have

by (1.1), (1.2), and (1.3), the equation

$$(2.2) \qquad R[\varphi_n + t\Psi] = \frac{A[\varphi_n, \varphi_n] + 2tA[\varphi_n, \Psi] + t^2 A[\Psi, \Psi]}{B[\varphi_n, \varphi_n] + 2tB[\varphi_n, \Psi] + t^2 B[\Psi, \Psi]} \quad .$$

The condition that this ratio be a minimum for $t = 0$ leads by elementary calculus to the requirement

$$(2.3) \qquad A[\varphi_n, \Psi] = \nu_n B[\varphi_n, \Psi] \quad .$$

In particular, we have for $m \geqslant n$

$$(2.3') \qquad A[\varphi_n, \varphi_m] = \nu_n B[\varphi_n, \varphi_m] = \begin{array}{lll} \nu_n & \text{if} & n = m \\ 0 & \text{if} & n \neq m \end{array} \quad .$$

Since this equation is symmetric in n and m it must hold for arbitrary n and m.

We observe now that we can obtain equality simultaneously in all m inequalities (1.17). For (1.13) yields, by virtue of (2.3′),

$$(2.3'') \qquad R[x_1\varphi_1 + \dots + x_n\varphi_n] = \frac{\nu_1 x_1^2 + \nu_2 x_2^2 + \dots + \nu_n x_n^2}{x_1^2 + x_2^2 + \dots + x_n^2} \quad .$$

Thus, if we choose $g_i = \varphi_i$ and (2.1), the determinantal equation (1.15) turns out to be simply

$$(\nu - \nu_1)(\nu - \nu_2)\dots(\nu - \nu_m) = 0$$

and so ν_i' becomes ν_i, and all the m inequalities (1.16) simultaneously become equations.

2.2. We consider as given the m linearly independent functions $g_1, g_2, \dots, g_m$. Their linear combinations (1.19) form an m-dimensional linear subspace S_m of Σ. Let us vary $x_1, x_2, \dots, x_m$, that is, let u vary in this subspace S_m, and let us seek the maximum of $R[u]$. If we assume (2.1), we easily conclude from (1.13), (2.1), (1.18), and (1.16) that

$$(2.4) \qquad \max_{u \in S_m} R[u] = \nu_m' \quad .$$

Take now the case considered in the foregoing Section 2.1 when for a suitable choice of the subspace S_m (namely $g_1 = \varphi_1$, $g_2 = \varphi_2, \dots, g_m = \varphi_m$) the inequalities (1.17) become equations. Considering the inequality (1.21) and (2.4), we see that

$$(2.5) \qquad \nu_m = \underset{S_m}{\mathrm{Min}}\ \underset{u \in S_m}{\max}\ R[u] \quad .$$

We used capital M in Min and lower case m in max to emphasize the difference between a variational extremum problem and an elementary extremum problem that is concerned only with a finite set of variables. The minimax definition (2.5) of the m^{th} eigenvalue ν_m is very different from the well known definition due to Courant [2, p. 113], which involves two successive variational extremum problems. In the case of finite quadratic forms (when Σ is a finite dimensional vector space), (2.5) is due to E. Fischer [4].

We may take the liberty to call an m-dimensional linear subspace S_m of Σ an m-dimensional "convoy". As the ships in a convoy must travel together, the speed of a convoy depends on the slowest ship. Even if we wish to make $R[u]$ as small as possible, (2.4) represents the best that S_m can do for $R[u]$ provided that S_m is considered as a convoy, as a set with solidary elements. If we seek the best m-dimensional convoy, (2.5) represents the solution of our problem. And so we may describe ν_m, the m^{th} eigenvalue of $R[u]$, as the "m-dimensional convoy-minimum of $R[u]$".

<u>2.3</u>. The present section deals with an elementary remark on the relation (1.17) that we shall need in the next section.

We consider finite sequences, each consisting of m real numbers, such as $\alpha_1, \alpha_2, \dots, \alpha_m$ or $\beta_1, \beta_2, \dots, \beta_m$. We say that the sequence $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_m}$ is a rearrangement of $\alpha_1, \alpha_2, \dots, \alpha_m$ if $i_1, i_2, \dots, i_m$ are different positive integers none of which exceeds m. Let $\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_m$ denote the rearrangement of $\alpha_1, \alpha_2, \dots, \alpha_m$ in <u>ascending order</u> so that

(2.6) $$\bar{\alpha}_1 \le \bar{\alpha}_2 \le \dots \le \bar{\alpha}_m \quad .$$

We say that $\alpha_1, \alpha_2, \dots, \alpha_m$ is <u>majorized</u> by $\beta_1, \beta_2, \dots, \beta_m$ if

(2.7) $$\bar{\alpha}_1 \le \bar{\beta}_1, \bar{\alpha}_2 \le \bar{\beta}_2, \dots, \bar{\alpha}_m \le \bar{\beta}_m \quad .$$

We express the fact that $\alpha_1, \dots, \alpha_m$ is majorized by $\beta_1, \dots, \beta_m$ in writing

(2.8) $$\alpha_1, \alpha_2, \dots, \alpha_m << \beta_1, \beta_2, \dots, \beta_m \quad .$$

Thus, as the ν and ν' are arranged in ascending order, $\nu_i = \bar{\nu}_i$, $\nu'_i = \bar{\nu}'_i$, and we may express (1.17) in the more concise form

(2.9) $$\nu_1, \nu_2, \dots \nu_m << \nu'_1, \nu'_2, \dots, \nu'_m \quad .$$

(a) <u>Assume that there are rearrangements</u> $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_m}$ <u>and</u> $\beta_{k_1}, \beta_{k_2}, \dots, \beta_{k_m}$ <u>of the</u> α <u>and of the</u> β, <u>respectively, such that</u>

(2.10) $$\alpha_{i_1} \le \beta_{k_1}, \alpha_{i_2} \le \beta_{k_2}, \dots, \alpha_{i_m} \le \beta_{k_m} \quad .$$

<u>Then</u> (2.7) <u>follows, and so the</u> α <u>are majorized by the</u> β.

Since a simultaneous permutation of the α and the β leaves obviously intact the essential hypothesis of this lemma (a), we can assume in proving it without loss of generality that the inequalities (2.10) are reduced to the form

(2.11) $$\bar{\alpha}_1 \le \beta_{\ell_1}, \bar{\alpha}_2 \le \beta_{\ell_2}, \dots, \bar{\alpha}_m \le \beta_{\ell_m}$$

where $\beta_{\ell_1}, \dots, \beta_{\ell_m}$ is a rearrangement of the β. If no pair of subscripts i and k exists such that

(2.12) $$i < k, \quad \beta_{\ell_i} > \beta_{\ell_k} \quad ,$$

(2.11) coincides already with the desired (2.7). If, however, there is such a pair i,k as described by (2.12), we infer from (2.11), in view of the ordering of the $\bar{\alpha}$ (see (2.6)) that

$$\bar{\alpha}_i \le \bar{\alpha}_k \le \beta_{\ell_k} < \beta_{\ell_i}$$

and so

$$\bar{\alpha}_i \leq \beta_{\ell_k}, \ \bar{\alpha}_k \leq \beta_{\ell_i} \quad .$$

Therefore, interchanging β_{ℓ_i} and β_{ℓ_k} we bring (2.11) one step nearer to the desired final situation (2.7) which we can thus attain in a finite number of similar steps. This proves (a).

(b) Suppose that the sequence $\alpha_1, \ldots, \alpha_m$ is majorized by the sequence $\beta_1, \ldots, \beta_m$ and there are two elements, α_i in the first sequence and β_k in the second, such that $\alpha_i = \beta_k$. If we delete α_i and β_k, the remaining α (n-1 in number) are still majorized by the remaining β (also n-1 in number).

If one of the defining inequalities (2.7) happens to be $\alpha_i \leq \beta_k$, the assertion is immediately obvious. Otherwise, there are subscripts i′ and k′, $i' \neq k$, $k' \neq i$ such that

(2.13) $$\alpha_i \leq \beta_{i'} \quad , \quad \alpha_{k'} \leq \beta_k$$

occur in (2.7). It follows, however, that

(2.14) $$\alpha_{k'} \leq \beta_k = \alpha_i \leq \beta_{i'} \quad .$$

Now, by virtue of lemma (a), we may rearrange the β, and, using (2.14) we may replace (2.13) by

$$\alpha_{k'} \leq \beta_{i'} \quad , \quad \alpha_i \leq \beta_k \quad .$$

This brings us back to the immediately obvious case and proves (b). There is no need to formulate the obvious extension of (b) to the deleting of several pairs, each of which consists of an α and a β equal to that particular α. We shall need, however, this extension in the next section.

2.4. Let us assume that those eigenfunctions of R[u] that belong to eigenvalues equal to the same number ν form a vector space of finite dimension. (This is so in the most usual problems of Mathematical Physics.)

If this dimension is ℓ, we say sometimes that ν is an eigenvalue of multiplicity ℓ, or (better) that ℓ eigenvalues coincide with ν. (In fact, ν is the common solution of ℓ different extremal problems, see Section 1.1 or (2.5).)

We introduce, however, a further, more special, assumption. Let p denote the set of variables (the point) on which the functions of the class Σ depend. We assume that there is an operation J of period 2 (J^2 is the identity, J is an "involution") that transforms the point p into $\tilde{p}$, so that

$$Jp = \tilde{p} \quad , \quad J\tilde{p} = J(Jp) = p \quad .$$

We assume that the involution J transforms every function u(p) of the class Σ into a function $u(\tilde{p})$ of the same class, and that

$$A[u(\tilde{p}),v(\tilde{p})] = A[u(p),v(p)] \quad , \quad B[u(\tilde{p}),v(\tilde{p})] = B[u(p),v(p)] \quad . \tag{2.15}$$

(<u>Example</u>. Let p stand for the point (x,y,z), $\tilde{p}$ for (-x,-y,-z), D for a bounded domain symmetric with respect to the origin, and let the class C consist of the functions having two continuous derivatives within D, continuous within, and on the boundary, of D, and vanishing on the boundary. We set

$$A[u,v] = \iiint (u_x v_x + u_y v_y + u_z v_z)\,dx\,dy\,dz \quad ,$$
$$B[u,v] = \iiint uv\,dx\,dy\,dz \quad ;$$

both integrals are extended over D. The conditions (2.15) are satisfied.

We could define $\tilde{p}$ differently, as (-x,y,z), or as (-x,-y,z), provided that D has the corresponding symmetry, with respect to the y,z-plane, or the z-axis, respectively.)

We call the function u(p) even or odd, according as it satisfies the first or the second equation

$$u(\tilde{p}) = u(p) \quad , \quad u(\tilde{p}) = -u(p) \quad . \tag{2.16}$$

We develop a few consequences of the above assumptions and definitions.

(a) <u>If u is odd and v is even</u>

$$A[u,v] = B[u,v] = 0 \quad .$$

In fact, in using the definition of even and odd functions and (1.2), we find

$$A[u(\tilde{p}),v(\tilde{p})] = A[-u(p),v(p)] = -A[u(p),v(p)] \quad .$$

In view of the first equation (2.15) the assertion concerning A follows, and that concerning B follows similarly.

(b) <u>If $\varphi(p)$ is an eigenfunction of $R[u]$ that belongs to the eigenvalue ν, also $\varphi(\tilde{p})$ is an eigenfunction belonging to the same eigenvalue ν.</u>

In fact, $\varphi(p)$, as an eigenfunction, must be obtained at a certain step of the recursive process described in Section 1.1. If this step is the n^{th}, $\varphi(p) = \varphi_n(p)$, $\nu = \nu_n$. Yet, in view of (2.15), we can substitute in the defining process

$$\varphi_1(\tilde{p}), \varphi_2(\tilde{p}), \ldots, \varphi_n(\tilde{p})$$

for

$$\varphi_1(p), \varphi_2(p), \ldots, \varphi_n(p)$$

without altering the corresponding eigenvalues

$$\nu_1, \nu_2, \ldots, \nu_n \quad .$$

This proves (b).

(c) <u>If ν is an eigenvalue of multiplicity ℓ, the eigenfunctions belonging to ν have a basis that consists of j even and k odd functions, $j+k = \ell$, $j \geqslant 0$, $k \geqslant 0$.</u>

Let $\varphi(p)$ be an eigenfunction that belongs to ν. By (b), $\varphi(\tilde{p})$ is also an eigenfunction that belongs to ν. Therefore, $\varphi(p) + \varphi(\tilde{p})$ is either identically 0, or, except for a constant factor, an even eigenfunction belonging to ν. Similarly, $\varphi(p) - \varphi(\tilde{p})$ is either identically 0, or an odd eigenfunction belonging to ν. Yet

$$(2.17) \qquad \varphi(p) = \tfrac{1}{2}[\varphi(p) + \varphi(\tilde{p})] + \tfrac{1}{2}[\varphi(p) - \varphi(\tilde{p})]$$

and both terms cannot vanish on the right-hand side. Thus, there exist either even eigenfunctions, or odd eigenfunctions, or both kinds exist. The even eigenfunctions form a linear subspace and possess a basis consisting of j functions, and the odd eigenfunctions possess a basis consisting of k functions. (If there are no eigenfunctions of one or the other kind, j or k may be 0, respectively.) At any rate, as (2.17) shows, any eigenfunction belonging to ν is a linear combination of these $j+k$ basis functions. These $j+k$ functions are obviously linearly independent, since only the identically vanishing function can be both odd and even, and so $j+k = \ell$. This proves (c).

We can describe the situation viewed in (c) and its proof, by saying that there are j even eigenvalues and k odd eigenvalues coinciding with ν; we call, of course, an eigenvalue odd or even according as the eigenfunction associated with it is odd or even. Under the assumptions of the present section, namely (2.15), we can label each single eigenvalue as even or odd.

(d) Assume that the functions $g_1, g_2, \ldots, g_m$ considered in Section 1.2 are even and that a_{ik} and b_{ik} are given by (2.1). Then the roots $\nu_1', \nu_2', \ldots, \nu_m'$ of the characteristic equation (1.15) majorize not only the first m eigenvalues, but the first m EVEN eigenvalues.

It is also true: "If the functions $g_1, g_2, \ldots, g_m$ are chosen as odd, $\nu_1', \nu_2', \ldots, \nu_m'$ majorize the first odd eigenvalues" and the method of proof that will be used applies equally to both cases.

Let n denote the number of those odd eigenvalues that are not greater than the m^{th} even eigenvalue, and let $g_{m+1}, g_{m+2}, \ldots, g_{m+n}$ denote the odd eigenfunctions belonging to these n odd eigenvalues. Apply the theorem of Section 1.2 to the m+n functions $g_1, \ldots, g_m, g_{m+1}, \ldots, g_{m+n}$ (instead of the

m functions $g_1,\ldots,g_m$.) Applying (a) to (2.1), we see that there are two m by n rectangles in the characteristic determinant (see (1.15)) one in the north-east corner, the other in the south-west corner, all elements in which vanish. Thus, the determinant is a product of two determinants, one of order m, the other of order n. The latter has, in view of (1.9) and (2.2), non-vanishing elements only in its main diagonal and yields as roots precisely the n odd eigenvalues associated with $g_{m+1},\ldots,g_{m+n}$. By the theorem of Section 1.2, the m+n roots of the determinant majorize the m+n first eigenvalues of R[u]. The n odd eigenvalues occur both in the majorizing, and in the majorized, set. If we delete these n odd eigenvalues, only the m roots corresponding to $g_1,g_2,\ldots,g_m$ remain in the majorizing set and only the m even eigenvalues remain in the majorized set. Using (b) of Section 2.3, we attain the assertion (d).

It does not diminish the interest of the foregoing proof that some more or less definite statement approaching (d) has often been tacitly assumed in the computation of eigenvalues.

Chapter III

Eigenvalue Problems

1. The Convexity Theorem.

1.1. In this chapter we shall combine Poincaré's theorem with the method of transplanting extremal functions. We will obtain various results on the dependence of eigenvalues on their domain of definition. We formulate an eigenvalue problem which is general enough to cover various important particular cases, but we do not intend to complicate our presentation by aiming at the greatest possible generality.

Let D be a fixed finite domain in the (x,y)-plane which is bounded by a smooth curve C; let s be the arc length on C, counted from some given point on C. We introduce two continuous non-negative functions on C

(1.1) $$h(s;\tau) \geqq 0 \quad , \quad k(s;\tau) \geqq 0$$

which depend on an additional real parameter τ whose role will become clear in the sequel. Let further

(1.2) $$p(x,y;\tau) > 0 \quad , \quad q(x,y;\tau) > 0 \quad , \quad r(x,y;\tau) > 0$$

be three continuous positive functions in D which also depend on the same parameter τ. Finally, let $\sum$ be the class of all functions $\Phi(x,y)$ which are continuously differentiable in D and are continuous in D+C. We define in $\sum$ the functional

(1.3) $$R_\tau[\Phi] = \frac{\int_C \Phi^2 \sqrt{h\,dx^2 + k\,dy^2} + \iint_D [p\Phi_x^2 + q\Phi_y^2]dx\,dy}{\iint_D r\,\Phi^2\,dx\,dy} ,$$

which still depends on the parameter τ. In the notation of Section (II, 1.1), we have

(1.4) $$A[u,v] = \int_C uv\sqrt{h\,dx^2 + k\,dy^2} + \iint_D [p\,u_x v_x + q\,u_y v_y]dx\,dy$$

and

(1.5) $$B[u,v] = \iint_D r\,uv\,dx\,dy \quad .$$

We define a set of eigenfunctions $u_\nu(x,y;\tau)$ and eigenvalues $\lambda_\nu(\tau)$ for the functional $R_\tau[\Phi]$ by the sequence of minimum problems described in Section (II, 1.1). The existence of extremal functions is guaranteed by the general calculus of variations and it is also well known that the ν^{th} eigenfunction u_ν satisfies the differential equation

(1.6) $$\frac{\partial}{\partial x}\left(p\frac{\partial u_\nu}{\partial x}\right) + \frac{\partial}{\partial y}\left(q\frac{\partial u_\nu}{\partial y}\right) + \lambda_\nu\, r u_\nu = 0$$

with the natural boundary condition on C

(1.7) $$p\frac{\partial u_\nu}{\partial x}\cos(n,x) + q\frac{\partial u_\nu}{\partial y}\cos(n,y) = u_\nu\sqrt{h\cos^2(n,y) + k\cos^2(n,x)} \quad ,$$

where n is the interior normal.

In view of the conditions (II, 1.9) imposed on the extremum function, we have the orthonormalization conditions

(1.8) $$B[u_\mu,u_\nu] = \delta_{\mu\nu}$$

where we denote as usual by $\delta_{\mu\nu}$ the Kronecker symbol

(1.9) $$\delta_{\mu\nu} = \begin{cases} 0 & \text{if } \mu \neq \nu \\ 1 & \text{if } \mu = \nu \end{cases} \quad .$$

On the other hand, we derive from the general statement (II, 2.3′) and (1.8) the set of equations

(1.10) $$A[u_\mu,u_\nu] = \lambda_\mu\delta_{\mu\nu} \quad .$$

We introduced the eigenfunctions u_ν and their eigenvalues λ_ν by a sequence of extremum problems and showed that they satisfied the linear partial differential equation (1.6) with the linear homogeneous boundary condition (1.7). It can be shown, conversely, that the system (1.6), (1.7) has only a countable set of eigenvalues λ_ν which tend to infinity with ν and that this set coincides with the set of extremum values of the functional $R_\tau[\Phi]$. In most applications, one is interested in solutions of

the differential system and the extremum problems considered are introduced for methodological reasons.

1.2. We wish now to make assumptions regarding the dependence of the functions (1.1) and (1.2) on the parameter τ. We put

$$\lambda_\nu = \lambda_\nu(\tau) \quad , \tag{1.11}$$

and are going to draw conclusions on the dependence of the eigenvalues upon the parameter. Let us consider at first:

Hypothesis I. h, k, p, and q *are continuously differentiable and convex from above in dependence on* τ. r *is independent of* τ.

We want to prove the

Conclusion. *The sum* $\sum_{\nu=1}^{N} \lambda_\nu(\tau)$ *is for arbitrary integer* N *convex from above as a function of* τ.

In fact, let us write for short $h = h(s;\tau)$, $h_o = h(s;\tau_o)$, etc. where τ_o is an arbitrary but fixed value of the parameter. By our hypothesis, we have at each point of C:

$$h \leq h_o + (\tau - \tau_o)h_1 \quad , \quad h_1 = \frac{\partial}{\partial \tau} h(s;\tau)\Big|_{\tau = \tau_o} \tag{1.12}$$

and three analogous inequalities for k, p, and q. Observe that by the inequality between geometric and arithmetic means, we have

$$\sqrt{(h_o dx^2 + k_o dy^2)(h\,dx^2 + k\,dy^2)} \leq \frac{h_o + h}{2}\, dx^2 + \frac{k_o + k}{2}\, dy^2 \tag{1.13}$$

$$\leq (h_o + \frac{\tau - \tau_o}{2} h_1)dx^2 + (k_o + \frac{\tau - \tau_o}{2} k_1)dy^2 \quad .$$

We define now the functional $R'_{\tau,\tau_o}[\Phi]$ which is obtained from $R_\tau[\Phi]$ if we replace

$$p \text{ by } p_o + (\tau - \tau_o)p_1 \quad , \quad q \text{ by } q_o + (\tau - \tau_o)q_1 \tag{1.14}$$

and

$$\sqrt{h\,dx^2 + k\,dy^2} \text{ by } (h_o dx^2 + k_o dy^2)^{-1/2}[(h_o + \frac{\tau-\tau_o}{2}h_1)dx^2 + (k_o + \frac{\tau-\tau_o}{2}k_1)dy^2] \ . \tag{1.15}$$

In view of the inequality (1.13) and the inequalities for p and q which are analogous to (1.12), we can assert that for each Φ of the class Σ we have

(1.16) $$R_\tau[\Phi] \leq R'_{\tau,\tau_o}[\Phi] \quad .$$

We may put

(1.17) $$R'_{\tau,\tau_o}[\Phi] = \frac{A'_{\tau,\tau_o}[\Phi,\Phi]}{B[\Phi,\Phi]} \quad .$$

Let $\Phi_1, \Phi_2, \ldots, \Phi_N$ be the N first eigenfunctions belonging to $R_{\tau_o}[\Phi]$. We form

(1.18) $$u = x_1\Phi_1 + x_2\Phi_2 + \ldots + x_N\Phi_N$$

and have by (1.16), (1.8), and (1.10) the inequality

(1.19) $$R_\tau[u] \leq \frac{\sum_1^N \sum_1^N a_{ik}x_i x_k}{\sum_1^N x_i^2}$$

with

(1.20) $$a_{ik} = A'_{\tau,\tau_o}[\Phi_i, \Phi_k] \quad .$$

We denote by $\lambda'_1, \ldots, \lambda'_N$ the eigenvalues of the symmetric matrix $((a_{ik}))$, arranged in increasing order of magnitude. According to Poincaré's theorem, we have

(1.21) $$\lambda_\nu \leq \lambda'_\nu \qquad \nu = 1,2,\ldots,N \quad .$$

Since the sum of the λ'_ν equals the trace of the matrix $((a_{ik}))$, we derive from the definition of R'_{τ,τ_o}:

(1.22) $$\sum_{\nu=1}^N \lambda_\nu \leq \int_C \frac{(h_o + \frac{\tau-\tau_o}{2} h_1)dx^2 + (k_o + \frac{\tau-\tau_o}{2} k_1)dy^2}{\sqrt{h_o dx^2 + k_o dy^2}} \cdot \sum_{\nu=1}^N \Phi_\nu^2$$

$$+ \iint_D \left\{ [p_o + (\tau-\tau_o)p_1] \sum_{\nu=1}^N \left(\frac{\partial \Phi_\nu}{\partial x}\right)^2 + [q_o + (\tau-\tau_o)q_1] \sum_{\nu=1}^N \left(\frac{\partial \Phi_\nu}{\partial y}\right)^2 \right\} dx\,dy \quad .$$

This inequality has the form

$$(1.23)\qquad \sum_{\nu=1}^{N} \lambda_\nu(\tau) \leq \sum_{\nu=1}^{N} \lambda_\nu(\tau_o) + (\tau - \tau_o) a_N \quad .$$

The geometric interpretation of this result is obvious: If we plot the function $y = \sum_{\nu=1}^{N} \lambda_\nu(\tau)$ against the variable τ, the curve lies always below the straight line $y = y_o + (\tau - \tau_o) a_N$, passing through the curve point (τ_o, y_o). Thus, the curve has at every point a supporting line and is convex from above. This proves our assertion.

1.3. We find from (1.22) the slope of the supporting line

$$(1.24)\qquad a_N = \frac{1}{2} \int_C \frac{h_1 dx^2 + k_1 dy^2}{\sqrt{h_o dx^2 + k_o dy^2}} \sum_{\nu=1}^{N} \Phi_\nu^2 + \iint_D \left[p_1 \sum_{\nu=1}^{N} \left(\frac{\partial \Phi_\nu}{\partial y}\right)^2 + q_1 \sum_{\nu=1}^{N} \left(\frac{\partial \Phi_\nu}{\partial y}\right)^2\right] dx\, dy \quad .$$

Let us suppose that for some given value of τ the N first eigenvalues $\lambda_\nu(\tau)$ are all different. In this case, it can be shown that the $\lambda_\nu(\tau)$ have derivatives with respect to τ if the coefficient functions h, k, p, and q are continuously differentiable functions of τ. Clearly, we find

$$(1.25)\qquad \frac{d\lambda_\nu(\tau)}{d\tau} = \frac{1}{2} \int_C \frac{h_1 dx^2 + k_1 dy^2}{\sqrt{h_o dx^2 + k_o dy^2}} \Phi_\nu^2 + \iint_D \left[p_1 \left(\frac{\partial \Phi_\nu}{\partial x}\right)^2 + q_1 \left(\frac{\partial \Phi_\nu}{\partial y}\right)^2\right] dx\, dy \quad .$$

In order to discuss the case of a multiple eigenvalue let us suppose that λ_N has the two eigenfunctions Φ_N and Φ_{N+1}. There is an ambiguity which eigenfunction should be taken in (1.24) and we may use as N^{th} eigenfunction as well the linear combination

$$(1.26)\qquad \Psi = \alpha \Phi_N + \beta \Phi_{N+1} \quad , \quad \alpha^2 + \beta^2 = 1 \quad .$$

This leads to

$$(1.27)\qquad a_N = a_{N-1} + \alpha^2 K + 2\alpha\beta L + \beta^2 M$$

where K, L, and M are easily computed from (1.24) and (1.26). Except for the singular case $L = 0$, $K = M$ we have therefore an infinity of supporting lines; the minimum and maximum slope of this family can be easily determined from (1.27). Thus, the curve $y(\tau)$ will, in general, have a corner point if the last eigenvalue in the sum is degenerate and possesses an eigenfunction Φ_{N+1}.

An extension of these considerations to the case of a greater degeneracy of the eigenvalue λ_N is obvious. We will, in general, find a corner for the above curve, due to the fact that the freedom of choice for Φ_N leads to supporting lines with different slopes.

1.4. In this section, we shall make

Hypothesis II. h, k, p, and q are independent of τ and r is continuously differentiable and convex from below in τ.

Conclusion. The sum $\sum_{\nu=1}^{N} \lambda_\nu(\tau)^{-1}$ is convex from below in τ for arbitrary choice of the integer N.

The proof is the same as in the previous case. We have by assumption

$$(1.28) \qquad r > r_o + (\tau - \tau_o) r_1 \quad , \quad r_1 = \frac{\partial r}{\partial \tau}\Big|_{\tau = \tau_o} \quad ,$$

if $r = r(x,y;\tau)$, $r_o = r(x,y;\tau_o)$. We replace in $R_\tau[\Phi]$ the function r in the denominator by $r_o + (\tau - \tau_o)r_1$ and obtain so a functional $R_{\tau,\tau_o}[\Phi]$ for which holds by (1.28):

$$(1.29) \qquad R_\tau[\Phi] \leq R'_{\tau,\tau_o}[\Phi] = \frac{A[\Phi,\Phi]}{\iint\limits_D [r_o + (\tau - \tau_o)r_1]\Phi^2 dx\,dy}$$

for arbitrary Φ of class Σ.

Let now $\Phi_\nu(x,y)$ $(\nu = 1,2,\ldots,N)$ be the N first eigenfunctions of $R_{\tau_o}[\Phi]$ and denote their corresponding eigenvalues by $\lambda_\nu^{(o)}$. We put

$$(1.30) \qquad u = \frac{x_1}{\sqrt{\lambda_1^{(o)}}}\Phi_1 + \frac{x_2}{\sqrt{\lambda_2^{(o)}}}\Phi_2 + \dots + \frac{x_N}{\sqrt{\lambda_N^{(o)}}}\Phi_N \quad .$$

By (1.8), (1.10), and (1.29), we have the inequality

$$(1.31) \qquad R_\tau[u] \leq \frac{x_1^2 + x_2^2 + \dots + x_N^2}{\sum\sum b_{ik}x_i x_k}$$

with

$$(1.32) \qquad b_{ik} = \iint_D [r_o + (\tau - \tau_o)r_1] \frac{\Phi_i \Phi_k}{\sqrt{\lambda_i^{(o)}\lambda_k^{(o)}}} \, dx\, dy \quad .$$

We denote by $\lambda_1', \dots, \lambda_N'$ the eigenvalues of the right-hand side quadratic ratio in (1.31) and have by Poincaré's theorem:

$$(1.33) \qquad \lambda_\nu \leq \lambda_\nu' \qquad \nu = 1,2,\dots,N \quad .$$

The λ_ν' are defined by the secular equation (II, 1, 15) with $a_{ik} = \delta_{ik}$ and the b_{ik} defined in (1.32). This equation may be brought into the form

$$(1.34) \qquad \left| b_{ik} - (\lambda')^{-1}\delta_{ik} \right| = 0 \quad .$$

Thus, we have

$$(1.35) \qquad \sum_{\nu=1}^{N} \frac{1}{\lambda_\nu'} = \sum_{\nu=1}^{N} b_{\nu\nu} = \iint_D [r_o + (\tau - \tau_o)r_1] \sum_{\nu=1}^{N} \frac{\Phi_\nu^2}{\lambda_\nu^{(o)}} \, dx\, dy \quad ,$$

and by (1.33)

$$(1.36) \qquad \sum_{\nu=1}^{N} \frac{1}{\lambda_\nu(\tau)} \geq \sum_{\nu=1}^{N} \frac{1}{\lambda_\nu(\tau_o)} + (\tau - \tau_o)\iint_D r_1 \sum_{\nu=1}^{N} \frac{\Phi_\nu^2}{\lambda_\nu(\tau_o)} \, dx\, dy \quad .$$

Thus, we have shown that the curve $y_N(\tau) = \sum_{\nu=1}^{N} \lambda_\nu(\tau)^{-1}$ has at every point a supporting line which lies entirely under it and this proves our statement. The slope of the supporting line is

$$(1.37) \qquad a_N = \iint_D r_1 \sum_{\nu=1}^{N} \frac{\Phi_\nu^2}{\lambda_\nu(\tau)} \, dx\, dy \quad .$$

This formula permits us again to calculate the derivatives $\frac{d}{d\tau}\lambda_\nu(\tau)$ if they exist and to discuss the corners of the curve $y_N(\tau)$ if degeneracy of eigenvalue occurs.

It should be observed that we have to assume in this section that the functional $A[\Phi, \Phi]$ is positive-definite in the class Σ. In fact, we operated with the reciprocals of the eigenvalues and the inequality (1.36) becomes meaningless if $\lambda_1(\tau) = 0$.

2. <u>Application to Stretching of Domains</u>.

<u>2.1</u>. In the preceding sections, we considered a variational problem in which the coefficients depended upon a parameter τ. We use the method of transplanting the extremal function insofar as we estimated the minimum for a given value τ by using the correct extremal function for τ_0 as a comparison function. In the following sections, we want to consider fixed variational problems with respect to domains which depend on a parameter τ. By referring the variable domains back to one fixed domain we will reduce the problem to one of the previous form, namely a variational problem for a fixed domain with coefficients depending upon a parameter τ. Thus, the results of the previous sections will become applicable to the problem of varying domains.

We shall study the minimum problem for the functional

$$R[\Phi] = \frac{k \int_C \Phi^2 ds + \iint_D (\nabla \Phi)^2 dx\, dy}{\iint_D \Phi^2 dx\, dy} \qquad (2.1)$$

The corresponding eigenfunctions $u_\nu(x,y)$ will satisfy the Euler-Lagrange equations

$$\Delta u_\nu + \lambda_\nu u_\nu = 0 \qquad (2.2)$$

with the natural boundary conditions on C

$$\frac{\partial u_\nu}{\partial n} = k u_\nu \qquad (2.3)$$

Clearly, the eigenvalues λ_ν depend on the parameter k. From Poincaré's theorem, we derive the result:

The eigenvalues $\lambda_\nu(k)$ increase monotonically with k. From the theorem of Section 1.2, we find further:

<u>Theorem</u> 1. The sum $\sum_{\nu=1}^{N} \lambda_\nu(k)$ is for arbitrary integer N convex from above as a function of k^2.

We may apply (1 24) in order to find the derivative of the above sum whenever it exists; we have

(2.4)
$$\frac{d}{dk} \sum_{\nu=1}^{N} \lambda_\nu(k) = \int_C \sum_{\nu=1}^{N} u_\nu^2 ds$$

if u_ν is the ν^{th} eigenfunction to the value k of the parameter.

If $k \to \infty$, the eigenvalues $\lambda_\nu(k)$ tend to limit values λ_ν which correspond to the eigenvalue problem

(2.5)
$$\Delta u + \lambda_\nu u = 0 \quad , \quad u = 0 \quad \text{on } C \quad ,$$

the problem of the vibrating membrane spanned over D and clamped in at C.

<u>2.2</u>. We consider the family D(t) of domains which are obtained from a given domain $D = D(1)$ by the one-sided stretching in the x-direction

(2.6)
$$x' = tx \qquad y' = y \quad .$$

The extremum problem for the functional $R[\Phi]$ over a domain D(t) may be expressed as the analogous problem for the functional

(2.7)
$$R_t[\Phi] = \frac{k \int_C \Phi^2 \sqrt{dx^2 + t^{-2} dy^2} + \iint_D [t^{-2} \Phi_x^2 + \Phi_y^2] dx\,dy}{\iint_D \Phi^2 \, dx\,dy}$$

and with respect to the original domain $D = D(1)$.

We put $t^{-2} = \tau$ and observe that the hypothesis I of the Section 1.2 is fulfilled. We may thus draw the following conclusion:

Theorem 2. If the domains $D(t)$ are obtained from $D = D(1)$ by one-sided stretching with the parameter t, the sums $\sum_{\nu=1}^{N} \lambda_\nu(k)$ are convex from above as functions of the parameter $\tau = t^{-2}$ for arbitrary fixed values N and k.

From (1.24) we calculate

$$\frac{d}{d\tau} \sum_{\nu=1}^{N} \lambda_\nu(k) = \frac{k}{2} \int_C \frac{dy^2}{\sqrt{dx^2 + \tau^2 dy^2}} \sum_{\nu=1}^{N} u_\nu^2 + \iint_D \sum_{\nu=1}^{N} \left(\frac{\partial u_\nu}{\partial x}\right)^2 dx\,dy . \tag{2.8}$$

This formula shows that the sums increase monotonically with the variable τ.

We consider, in particular, a rectangle with sides of length a, b and parallel to the x- and y-axis respectively. The eigenvalue problem (2.2), (2.3) with $k = 0$ has the eigenfunctions

$$u_\nu = \cos \frac{\ell\pi}{a} x \cos \frac{m\pi}{b} y \tag{2.9}$$

with the eigenvalues

$$\lambda_\nu = \pi^2\left(\frac{\ell^2}{a^2} + \frac{m^2}{b^2}\right) , \quad \ell \geq 0, \quad m \geq 0 . \tag{2.10}$$

Under a stretching in the x-direction with parameter t, we thus obtain

$$\sum_{\nu=1}^{N} \lambda_\nu = \tau \frac{\pi^2}{a^2} \sum \ell^2 + \frac{\pi^2}{b^2} \sum m^2 \tag{2.11}$$

where the summation is to be extended over all pairs (ℓ, m) which lead to the N first eigenvalues. As long as no λ_ν is degenerate the coefficients in the right side linear function remain unchanged and the left side sum is a linear function of τ. This result shows that the convexity statement of the preceding theorem is the best possible statement with respect to the second derivative of the sum, considered. If λ_N is degenerate for $\tau = \tau_o$ and we have the two representations

$$\lambda_N = \pi^2\left(\tau_o \frac{\ell_1^2}{a^2} + \frac{m_1^2}{b^2}\right) = \pi^2\left(\tau_o \frac{\ell_2^2}{a^2} + \frac{m_2^2}{b^2}\right) \tag{2.12}$$

with $\ell_2 < \ell_1$, we have to use the pair (ℓ_1, m_1) in the formula (2.11) as long as $\tau \leq \tau_o$ but have to replace it by (ℓ_2, m_2) for $\tau > \tau_o$. Thus, the slope of the line (2.11) suffers a discontinuity at the point of degeneracy τ_o and it is clear that its value decreases. This agrees with the convexity theorem, which covers, however, much more complicated situations than the one discussed here.

Similarly, the eigenvalue problem (2.5) of the membrane leads in the case of the rectangle to the eigenvalues

$$\lambda_\nu = \pi^2\left(\frac{\ell^2}{a^2} + \frac{m^2}{b^2}\right) \quad , \quad \ell > 0 \quad , \quad m > 0 \tag{2.13}$$

and an analogous situation arises.

2.3. The value of the convexity theorem of the preceding section comes from the fact that it permits to estimate the eigenvalues of a whole class of domains D(t) if the corresponding eigenvalues are known for a few domains of the class only. We want to illustrate this fact by a special case which will be discussed in detail.

Let D be the square with diagonals of length π; we assume that the coordinate axes coincide with the diagonals of D and we consider the family of rhombi D(t) obtained from D by one-sided stretching in the x-direction with parameter t. The eigenvalue problem to be studied for this class of domains is

$$\Delta u + \lambda u = 0 \quad , \quad \frac{\partial u}{\partial n} = 0 \quad . \tag{2.14}$$

The lowest eigenvalue of this problem is always $\lambda = 0$ corresponding to a constant eigenfunction. We want to study the value of $\mu = \lambda_2$ for this problem and for the rhombi D(t).

By virtue of the convexity theorem of Section 2.2 with $k = 0$, $N = 2$, the function μ depends convexly upon $\tau = t^{-2}$. We shall compute now $\mu(\tau)$ for

the value $\tau = 1$, that is for the square, and determine two supporting lines through this point to the curve $\mu(\tau)$. This will lead to an upper estimate for all points of the curve.

Since the sides of the square D have the length $a = b = \frac{\pi}{2}\sqrt{2}$, we obtain from (2.10)

$$\mu(1) = 2 \quad . \tag{2.15}$$

This eigenvalue can be obtained by choosing $\ell = 1$, $m = 0$, or $\ell = 0$, $m = 1$ and is, therefore, of degeneracy 2. We choose as eigenfunctions

$$f = \frac{2\sqrt{2}}{\pi} \cos x \sin y \qquad g = \frac{2\sqrt{2}}{\pi} \sin x \cos y \tag{2.16}$$

which obviously satisfy (2.14) with $\lambda = 2$ and are orthonormalized. We can easily compute

$$\iint_D f_x^2 \, dx\, dy = \frac{1}{\pi^2}[\pi^2 - 8] \quad , \quad \iint_D g_x^2 \, dx\, dy = \frac{1}{\pi^2}[\pi^2 + 8] \tag{2.17}$$
$$\iint_D f_x g_x \, dx\, dy = 0 \quad .$$

We may now use any combination

$$u = \alpha f + \beta g \quad , \quad \alpha^2 + \beta^2 = 1 \tag{2.18}$$

as a normalized eigenfunction in obtaining a supporting line at the point $\tau = 1$, $\mu = 2$ by means of (1.24). We find for the slope

$$a = \iint_D \left(\frac{\partial u}{\partial x}\right)^2 dx\, dy \tag{2.19}$$

and by (2.17), (2.18)

$$a = \frac{1}{\pi^2}\left\{\alpha^2(\pi^2 - 8) + \beta^2(\pi^2 + 8)\right\} \quad . \tag{2.20}$$

The extreme supporting lines through the point considered have the slopes

$$a_1 = 1 + \frac{8}{\pi^2} \quad , \quad a_2 = 1 - \frac{8}{\pi^2} \quad . \tag{2.20 }$$

Thus, we have the inequalities

(2.21)
$$\mu(\tau) \leq \left(1+\frac{8}{\pi^2}\right)\tau + \left(1-\frac{8}{\pi^2}\right) \quad , \quad \text{if} \quad \tau \leq 1$$
$$\mu(\tau) \leq \left(1-\frac{8}{\pi^2}\right)\tau + \left(1+\frac{8}{\pi^2}\right) \quad , \quad \text{if} \quad \tau \geq 1$$

for the second eigenvalue of a rhombus whose diagonals have the length $\tau^{-1/2}\pi$ and π.

We shall now determine the value of $\mu(0)$ and the direction of the supporting line through this point. The convexity of the curve $\mu(\tau)$ will lead us then to another estimate valid for the entire curve. Since the first eigenfunction of the system (2.14) is a constant and each higher eigenfunction must be orthogonal to it, clearly if $u_2(x,y)$ is the second eigenfunction, we must have

(2.22)
$$\iint_D u_2 \, dx\, dy = 0 \quad .$$

We may characterize the second eigenvalue by the extremum problem

(2.23)
$$\mu = \min \frac{\iint_D (\nabla u)^2 \, dx\, dy}{\iint_D u^2 \, dx\, dy}$$

among all not identically vanishing, continuously differentiable functions in D+C which satisfy

(2.24)
$$\iint_D u \, dx\, dy = 0 \quad .$$

Consider now the rhombus whose diagonals lie on the x- and y- axis and have the length $t\pi$ and π, respectively. Consider an arbitrary function $u(x)$ which is odd and continuously differentiable in its single variable. Because of the symmetry of the rhombus in the y-axis, the conditon (2.24) is guaranteed. Using $u(x)$ as a comparison function in (2.23), we find

(2.25)
$$\mu \leq \frac{\iint_D u'(x)^2 \, dx\, dy}{\iint_D u^2 \, dx\, dy} \quad .$$

Using the geometry of the rhombus, we can simplify and obtain

$$\mu \leq \frac{\int_0^{\frac{\pi}{2}t} u'(x)^2 \left(\frac{\pi}{2} - \frac{x}{t}\right) dx}{\int_0^{\frac{\pi}{2}t} u(x)^2 \left(\frac{\pi}{2} - \frac{x}{t}\right) dx} \quad (2.26)$$

We introduce the new variable

$$\zeta = 1 - \frac{2x}{\pi t} \quad (2.27)$$

which runs from 0 to 1 as x runs from $\frac{\pi}{2}t$ to zero. Clearly, we can choose any continuously differentiable function $v(\zeta)$ defined in the interval $\langle 0,1 \rangle$ and vanishing at $\zeta = 1$ and define

$$u(x) = v\left(1 - \frac{2x}{\pi t}\right) \quad \text{for} \quad 0 \leq x \leq \frac{\pi}{2}t \quad (2.28)$$

and continue u(x) as an odd function into the second half of the rhombus. Each $v(\zeta)$ leads to an estimate:

$$\mu \leq \frac{\int_0^1 \zeta v'(\zeta)^2 d\zeta}{\int_0^1 \zeta v^2 d\zeta} \cdot \frac{4}{\pi^2 t^2} \quad (2.29)$$

We obtain the best estimate if we choose $v(\zeta)$ in such a way that the ratio on the right side of (2.29) becomes a minimum. This is an elementary problem of the calculus of variations; the Euler-Langrange equation for $v(\zeta)$ is

$$\zeta v'' + v' + \Lambda^2 \zeta v = 0 \quad (2.30)$$

with the boundary conditions

$$v'(0) = 0 \qquad v(1) = 0 \quad (2.31)$$

The first condition is the natural boundary condition for $\zeta = 0$ while the second condition was imposed by the requirement that u(x) be an odd function. Λ^2 is the minimum value of the ratio considered.

We derive from (2.30) that the best possible $v(\varsigma)$ has the form

(2.32) $$v(\varsigma) = J_o(\Lambda \varsigma)$$

where $J_o(x)$ is the Bessel function of order zero. In order to fulfill (2.31), we have to choose Λ as the first root of $J_o(x)$. We have

(2.33) $$\Lambda = 2.4048$$

and (2.29) may be put into the form

(2.34) $$\mu \leq \frac{4\Lambda^2}{\pi^2}\tau \quad , \quad \tau = t^{-2} \quad .$$

This estimate determines a straight line through the origin with slope $a = 2.3426$ under which the whole curve $\mu(\tau)$ has to lie. We utilized till now only the minimum definition (2.23) of the second eigenvalue. If we add now the convexity of the curve the following additional estimate arises. The curve $\mu(\tau)$ satisfies

(2.35) $$\mu(\tau) \geqslant 2\tau \qquad \text{for} \qquad 0 \leq \tau \leq 1$$

since the line 2τ is a secant of the curve.

The straight lines (2.21), (2.34), and (2.35) bound in the (μ,τ)-plane a narrow triangle inside of which the curve $\mu(\tau)$ has to lie as long as $0 \leq \tau \leq 1$. It is, on the other hand, sufficient to know the curve $\mu(\tau)$ for values $0 \leq \tau \leq 1$. In fact, if we perform a similarity on the domain D with eigenvalues λ_ν

(2.36) $$x' = \alpha x \qquad\qquad y' = \alpha y$$

we obtain a domain D' with eigenvalues $\alpha^{-2}\lambda_\nu$ as is easily seen from the system (2.14) by transplanting the eigenfunctions. In particular, the rhombus with diagonals $t\pi$, π goes over into another rhombus with diagonals π, $\frac{1}{t}\pi$ if we choose $\alpha = t^{-1}$. Hence, we have the functional equation

(2.37) $$\mu\left(\frac{1}{\tau}\right) = \frac{1}{\tau}\,\mu(\tau) \quad ,$$

which defines $\mu(\tau)$ for values $\tau > 1$ if $\mu(\tau)$ is already known for $\tau \leq 1$.

Combining (2.34) and (2.37) we find the estimate

$$(2.38) \qquad \mu(\tau) \leq \frac{4\Lambda^2}{\pi^2} = 2.3438$$

valid for all values of τ. Similarly, we can state that

$$(2.39) \qquad \mu(\tau) \geqslant 2 \quad \text{for} \quad \tau \geqslant 1 \quad .$$

These two estimates together with (2.21) delimit the curve $\mu(\tau)$ for $\tau \geqslant 1$. [Cf. Fig. I.]

2.4. We may use the artifice of the preceding section in order to obtain a general statement concerning the asymptotic behavior of the eigenvalues for the problem (2.14) as the parameter t of the stretching tends to zero or infinity. In fact, let $D = D(1)$ be given and let $f(x)$ be continuously differentiable in this domain and satisfy the condition

$$(2.40) \qquad \iint_D f(x)\,dx\,dy = 0 \quad .$$

If $D(t)$ is the domain which is obtained from D by the stretching (2.6) we define in it the function

$$(2.41) \qquad F(x) = f\left(\frac{x}{t}\right) \quad .$$

This function satisfies the condition (2.40) with respect to its domain of definition.

Hence, by the extremum definition (2.23), we may assert the inequality

$$(2.42) \qquad \mu(t) \leq \frac{\iint_D f'(x)^2\,dx\,dy}{\iint_D f(x)^2\,dx\,dy} \cdot \frac{1}{t^2}$$

for the first non-zero eigenvalue of the problem (2.14). Introducing again the variable $\tau = t^{-2}$, we have

$$(2.42') \qquad \mu(t) \leq K\tau$$

where K is an appropriate constant. This result shows that μ tends to zero with τ and is bounded from above by a linear function of τ if τ tends to infinity.

It is quite interesting to compare the lowest eigenvalue λ of the problem (2.5) in its asymptotic character with the eigenvalue μ. For this purpose, we consider two concentric rectangles with sides a, b, and A, B, respectively, such that the first lies in the domain D while the second rectangle contains this domain. From the minimum definition of λ follows immediately that this eigenvalue decreases if its domain of definition increases. Hence, we find the inequality between the lowest eigenvalues of the three nested domains, considered:

$$\pi^2\left(\frac{1}{A^2}+\frac{1}{B^2}\right) \leq \lambda \leq \pi^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right) \qquad . \tag{2.43}$$

If we subject now the entire plane to the stretching (2.6), the relation between the corresponding image domains remains unchanged such that (2.43) leads to the new inequality (with $\tau = t^{-2}$)

$$\pi^2\left(\frac{1}{A^2}\tau+\frac{1}{B^2}\right) \leq \lambda \leq \pi^2\left(\frac{1}{a^2}\tau+\frac{1}{b^2}\right) \qquad . \tag{2.44}$$

We see that $\lambda(t)$ does not tend to zero with τ. If we plot λ against τ, the curve will lie between the two straight lines given by (2.44).

It is not difficult to extend these considerations also to the case of higher eigenvalues; the natural tool is provided by Poincaré's theorem of Section II, 1.2.

2.5. Let us consider a domain D which appears from the origin O under an angle $\delta < 2\pi$. We may express the functional (2.1) of a function Φ in polar coordinates r, φ and have

$$R[\Phi] = \frac{k\int_C \Phi^2\sqrt{dr^2+r^2d\varphi^2} + \iint_D [\Phi_r^2 + \frac{1}{r^2}\Phi_\varphi^2]\, r\, dr\, d\varphi}{\iint_D \Phi^2\, r\, dr\, d\varphi} \qquad . \tag{2.45}$$

We subject now the whole plane to an angular stretching

(2.46) $\qquad r' = r \quad , \quad \varphi' = t\varphi \quad , \quad 0 < t < 2\pi\delta^{-1}$

as defined in Section I, 5.1. We obtain a sequence of domains $D(t)$ and corresponding functionals $R[\Phi]$ with respect to them. We may express these functionals also with respect to the original domain $D = D(1)$ by referring the independent variables back by means of (2.46). We obtain a sequence of functionals in the fixed domain D:

(2.47) $$R_t[\Phi] = \frac{k\int_C \Phi^2\sqrt{t^{-2}dr^2 + r^2 d\varphi^2} + \iint_D [\Phi_r^2 + \frac{t^{-2}}{r^2}\Phi_\varphi^2] r\,dr\,d\varphi}{\iint_D \Phi^2 r\,dr\,d\varphi} .$$

The eigenvalues of $R_t[\Phi]$ are the same as the eigenvalues of the functional $R[\Phi]$ with respect to the domain $D(t)$.

If we put again $\tau = t^{-2}$, we observe that Hypothesis I of Section 1.2 is fulfilled. Hence we proved:

Theorem 3. Under angular stretching with the parameter t, the sums $\sum_{\nu=1}^{N} \lambda_\nu(k)$ are convex from above as functions of $\tau = t^{-2}$ for arbitrary N and k.

We illustrate this result by the following application. Let D_a be the circular sector

(2.48) $\qquad 0 \le r \le 1 \quad , \quad 0 \le \varphi \le \frac{\pi}{a} \quad , \quad a > 1/2 \quad .$

The lowest eigenvalue of the membrane problem (2.5) for this domain belongs to the eigenfunction

(2.49) $\qquad u_1(r, \varphi) = J_a(\alpha r) \sin a\varphi$

where J_a is the Bessel function with index a and α is its first root. The eigenvalue corresponding to the function (2.49) is

(2.50) $\qquad \lambda_1 = \alpha^2 \quad .$

The first root α of J_a is an analytic function of the index

(2.51) $$\alpha = A(a) \quad .$$

Since the domain D_a belonging to a is obtained from D_1 by angular stretching with the parameter a^{-1}, we derive from Theorem 3:

The function $A(a)^2$ is convex from above as a function of a^2.

Thus, our convexity theorem for the eigenvalues of the membrane equation leads to a statement on the behavior of the roots of Bessel functions with continuously varying index. More results of this form can be easily obtained if we consider various values of N and k in the above theorem.

It is clear that similar considerations can be carried out for eigenvalue problems in more than two independent variables. We leave the formulation of the corresponding convexity theorems to the reader.

3. Application to Conformal Mapping.

3.1. We consider again a smoothly bounded finite domain D in the (x,y)-plane and a function $f(z)$, $z = x+iy$, which is regular analytic in the closure of D. Consider the conformal mapping

(3.1) $$z^* = z + tf(z)$$

which depends on a real parameter t. If t is small enough this mapping will be univalent and will carry the domain $D = D(0)$ into a domain $D(t)$.

We shall consider the membrane eigenvalue problem (2.5) with respect to the different domains $D(t)$ and will have to deal with the functional

(3.2) $$R[\Phi] = \frac{\iint (\nabla \Phi)^2 \, dx\, dy}{\iint \Phi^2 \, dx\, dy} \quad ,$$

considered for functions $\Phi(x,y)$ of the class Σ in $D(t)$ which vanish on the boundary $C(t)$. We may refer back the independent variables x, y in $D(t)$ into $D(0)$ by using the transformation formula (3.1). Since the Dirichlet integral in the numerator of $R[\Phi]$ is invariant under conformal transformation,

e arrive at the sequence of functionals

$$(3.3)\qquad R_t[\Phi] = \frac{\iint\limits_{D_o} (\nabla\Phi)^2\,dx\,dy}{\iint\limits_{D_o} r\,\Phi^2\,dx\,dy} \quad , \quad r = |1 + tf'(z)|^2 \quad .$$

These functions are to be considered now for the fixed domain $D = D(0)$. The eigenvalue problems connected with this functional lead to the differential system

$$(3.4)\qquad \Delta u + \lambda\,|1 + tf'(z)|^2\,u = 0 \qquad , \quad u = 0 \quad \text{on } C \quad .$$

Thus, instead of working with the fixed differential system (2.5) for the variable domain $D(t)$, we are using the variable system (3.4) for the fixed domain D. One advantage of this method of investigation is that we do not need to require that the transformation (3.1) be one-to-one in the complex plane but can admit all analytic mappings (3.1). We are, of course, essentially interested only in domains $D(t)$ which lie simply over the complex plane; but it is often useful to connect such a domain $D(t)$ with an initial domain $D = D(0)$ by a continuous deformation (3.1) which is not univalent for all intermediate values of t.

Since

$$(3.5)\qquad r = 1 + 2t\,\mathrm{Re}\left\{f'(z)\right\} + t^2|f'(z)|^2 \quad ,$$

we see that r is convex from below in t and that Hypothesis II of Section 1.4 holds. Hence, we can draw the conclusion:

<u>Theorem</u> 4. If $\lambda_\nu(t)$ are the eigenvalues of the system (3.4) ordered in increasing magnitude, then the sum $\sum_{\nu=1}^{N} \lambda_\nu(t)^{-1}$ is convex from below as a function of the parameter t.

The curve $y_N(t) = \sum_{\nu=1}^{N} \lambda_\nu(t)^{-1}$ has at every point a supporting line with the slope

$$(3.6)\qquad a_N = 2\iint\limits_{D} [\mathrm{Re}\left\{f'(z)\right\} + t\,|f'(z)|^2]\sum_{\nu=1}^{N} \lambda_\nu(t)^{-1} u_\nu^2\,dx\,dy \quad .$$

3.2. In the special case which we consider now, we can make a stronger statement than Theorem 4. Let Φ_ν ($\nu = 1,2,\ldots,N$) be the first N eigenfunctions of R_0 and $\lambda_\nu(0)$ the corresponding eigenvalues. We put

(3.7) $$u = \sum_{\nu=1}^{N} \frac{x_\nu}{\sqrt{\lambda_\nu(0)}} \Phi_\nu$$

and have

(3.8) $$R_t[u] = \frac{\sum_{\nu=1}^{N} x_\nu^2}{\sum\sum b_{ik} x_i x_k}$$

with

(3.9) $$b_{ik} = \iint_D |1 + tf'(z)|^2 \frac{\Phi_i \Phi_k}{\sqrt{\lambda_i(0)\lambda_k(0)}} \, dx\, dy \quad .$$

Just as in Section 1.4, we conclude from Poincaré's theorem

(3.10) $$\sum_{\nu=1}^{N} \lambda_\nu(t)^{-1} \geqslant \iint_D |1 + tf'(z)|^2 \sum_{\nu=1}^{N} \lambda_\nu(0)^{-1} \Phi_\nu^2 \, dx\, dy \quad .$$

We have by Schwarz's inequality

(3.11) $$\Big(\iint_D \mathrm{Re}\{f'(z)\} \sum_{\nu=1}^{N} \lambda_\nu(0)^{-1} \Phi_\nu^2 \, dx\, dy\Big)^2$$

$$\leq \iint_D \sum_{\nu=1}^{N} \lambda_\nu(0)^{-1} \Phi_\nu^2 \, dx\, dy \iint_D |f'(z)|^2 \sum_{\nu=1}^{N} \lambda_\nu(0)^{-1} \Phi_\nu^2 \, dx\, dy$$

$$\leq [Y_N(0)]^2 \iint_D |f'(z)|^2 \sum_{\nu=1}^{N} \lambda_\nu(0)^{-1} \Phi_\nu^2 \, dx\, dy \quad ,$$

where we put

(3.12) $$Y_N(t) = \Big[\sum_{\nu=1}^{N} \lambda_\nu(t)^{-1}\Big]^{1/2} \quad .$$

We derive from (3.10) by means of (3.11) the weaker inequality

(3.13) $$Y_N(t) \geqslant Y_N(0) + tY_N(0)^{-1} \iint_D \mathrm{Re}\{f'(z)\} \sum_{\nu=1}^{N} \lambda_\nu(0)^{-1} \Phi_\nu^2 \, dx\, dy \quad .$$

Thus, we have shown the existence of a supporting line to the curve $Y_N(t)$ at $t = 0$.

But from the above result follows the existence of a supporting line at every point. In fact, let z_o be obtained from z by a mapping (3.1) with the parameter t_o. We can express $z = F(z_o)$ by mapping $D(t_o)$ back into D; on the other hand, we can express the transformation (3.1) in the equivalent form

$$z^* = z_o + (t - t_o) f[F(z_o)] \quad . \tag{3.15}$$

Thus, using $t^* = t - t_o$ as a new parameter, we can repeat the preceding argument and establish a supporting line at the point $t = t_o$. Hence, we have proved:

<u>Theorem</u> 5. The function $[\sum_{\nu=1}^{N} \lambda_\nu(t)^{-1}]^{1/2}$ is convex from below as a function of the parameter t.

<u>3.3</u>. Let D be the unit circle $|z| < 1$; then all eigenfunctions of the membrane problem (2.5) have the form

$$u(x,y) = A_{n,m} J_n(k_{n,m} r) \begin{matrix} \cos n\varphi \\ \sin n\varphi \end{matrix} \tag{3.16}$$

where $J_n(x)$ is the Bessel function of order n and where $k_{n,m}$ is the m^{th} root of $J_n(x)$. The corresponding eigenvalues are

$$\lambda = k_{n,m}^2 \quad . \tag{3.17}$$

λ has only one eigenfunction if it has the form $k_{o,m}^2$ and is otherwise of degeneracy 2.

Consider now conformal transformation (3.1) with

$$f(z) = a_2 z^2 + a_3 z^3 + \ldots \quad . \tag{3.18}$$

The curve $y_N(t) = \sum_{\nu=1}^{N} \lambda_\nu(t)^{-1}$ has for $t = 0$ supporting lines with the slopes

$$a_N = 2 \iint_D \mathrm{Re}\{f'(z)\} \sum_{\nu=1}^{N} \lambda_\nu(0)^{-1} u_\nu^2 \, dx\, dy \quad . \tag{3.19}$$

If $\lambda_N \neq \lambda_{N+1}$, it is easily seen that

(3.20) $$\sum_{\nu=1}^{N} \lambda_\nu^{-1} u_\nu^2 = \sum \frac{J_n(k_{n,m} r)^2 A_{n,m}^2}{k_{n,m}^2}$$

and does not depend on the angular variable φ. From the fact that the harmonic function

(3.21) $$\mathrm{Re}\left\{f'(z)\right\} = \sum_{\nu=1}^{\infty} [\alpha_\nu r^\nu \cos \nu\varphi + \beta_\nu r^\nu \sin \nu\varphi]$$

has no term independent of φ, follows that $a_N = 0$. Since the curve $y_N(t)$ is convex from below and has a horizontal supporting line for $t = 0$, it follows that $y_N(t)$ has its minimum for $t = 0$.

Consider next the case that $\lambda_N = \lambda_{N+1}$; now we can choose the N^{th} eigenfunction u in various ways, namely in the form

(3.22) $$u_N = J_n(k_{n,m} r)[\alpha \cos n\varphi + \beta \sin n\varphi] \quad , \quad \alpha^2 + \beta^2 = 1 \quad .$$

Now, we have obviously

(3.23) $$\sum_{\nu=1}^{N} \lambda_\nu^{-1} u_\nu^2 = A(r) + B(r)(\alpha^2 \cos^2 n\varphi + \beta^2 \sin^2 n\varphi + 2\alpha\beta \cos n\varphi \sin n\varphi).$$

Inserting this representation into (3.19) and denoting

(3.24) $$\iint_D \mathrm{Re}\left\{f'(z)\right\} B(r) \cos^2 n\varphi \, dx\, dy = p$$

$$\iint_D \mathrm{Re}\left\{f'(z)\right\} B(r) \sin^2 n\varphi \, dx\, dy = q$$

$$\iint_D \mathrm{Re}\left\{f'(z)\right\} B(r) \sin 2n\varphi \, dx\, dy = 2s$$

we find an infinity of possible slopes

(3.25) $$a_N = 2(\alpha^2 p + \beta^2 q + 2\alpha\beta s) \quad .$$

Now clearly $p + q = 0$; put $\alpha = \cos\omega$, $\beta = \sin\omega$ and obtain

(3.26) $$a_N = 2[p \cos 2\omega + s \sin 2\omega] \quad .$$

Thus the slope a_N can range from $-2\sqrt{p^2 + s^2}$ to $+2\sqrt{p^2 + s^2}$ and the curve $y_N(t)$ has, in general, a corner at the point $t = 0$. Clearly, both branches of the curve rise from that point and since the curve is convex from below,

we see that in this case too $y_N(0)$ is the lowest point of the curve. Thus, we proved for every integer N, under the assumption that $f(z)$ has the form (3.18), the inequality

(3.27) $$y_N(0) \leq y_N(t)$$.

Let now D^* be an arbitrary domain which contains the origin $z = 0$ and can be obtained from the unit circle by a univalent mapping function

(3.28) $$k(z) = z + \sum_{\nu=2}^{\infty} a_\nu z^\nu$$;

in other words, we assume that D^* has the mapping radius 1 with respect to the point $z = 0$. We can deform the unit circle continuously into D^* by the transformation

(3.29) $$z^* = z + t[k(z) - z] \quad , \quad 0 \leq t \leq 1$$.

By virtue of (3.27), we can then assert that

(3.30) $$\sum_{\nu=1}^{N} (\lambda_\nu^*)^{-1} \geq \sum_{\nu=1}^{N} \lambda_{\nu,o}^{-1}$$,

where the right-hand side is the sum of the reciprocal eigenvalues for the unit circle

An arbitrary simply-connected domain D can be obtained from the unit circle by conformal mappings

(3.31) $$k(z) = a_o + a_1 z + \dots + a_n z^n + \dots$$

such that the center $z = 0$ of the circle goes into an arbitrarily prescribed point a_o of D. If we consider all possible choices of a_o. there exists a maximum value for a_1 . This value r_i is called the inner radius of D. We can assume that the mapping connected with this maximum coefficient transforms the origin into the origin; for otherwise, we achieve this by a translation of D. Next, we shrink D similar to itself in the ratio 1:r; each eigenvalue λ_ν in D goes over into the corresponding $\lambda_{\nu*} = \lambda_\nu r_i^2$ of the shrunk domain D^*. We map the unit circle into D^* by a map function

(3.28). Hence, we conclude from (3.30):

(3.32) $$\sum_{\nu=1}^{N} \lambda_\nu^{-1} \cdot r_i^{-2} \geqslant \sum_{\nu=1}^{N} \lambda_{\nu,o}^{-1} \quad .$$

Thus we have proved

<u>Theorem</u> 6. For $N = 1,2,\ldots$, the expression $\sum_{\nu=1}^{N} \lambda_\nu^{-1}\, r_i^{-2}$ has the minimum value if the domain D is a circle.

It is easily seen that the circle is the only minimum domain for Theorem 6 and the inequality (3.30). For we deduce from inequality (3.10) that the sum does actually increase with t except for the case in which $f'(z) \equiv 0$.

This fact leads to the following consequence. Let D be an arbitrary simply-connected domain containing the origin such that

(3.33) $$\iint_D \mathrm{Re}\left\{f'(z)\right\} \sum_{\nu=1}^{N} \lambda_\nu^{-1}\, u_\nu^2 \, dx\, dy = 0$$

for some N and all functions $f(z)$ analytic in D with $f\,(0) = 0$. Then D is necessarily a circle. In fact, from (3.33) and the convexity theorem 4, we can show that D leads to a minimum in Theorem 6 and by the uniqueness statement for the circle it follows that D must be a circle.

Chapter IV

Symmetry

1. Notation and Results.

1.1. We consider a bounded domain D in a plane referred to rectangular coordinates x,y and several functionals of D, namely A, I, P, K, k, and λ.

A is the area of D.

I is the polar moment of inertia of D with respect to its center of gravity.

P is the torsional rigidity of D.

K is the electrostatic capacity of D (conceived as a plate, or a cylindrical conductor with the base D and infinitesimal height).

k is only defined when D is doubly connected (ringshaped), bounded by an interior curve C_o and an exterior curve C_1. Consider the two infinite cylinders perpendicular to the (x,y)-plane that intersect it in the curves C_o and C_1, respectively. The capacity of the condenser formed by these two cylinders, per unit length, is called k.

λ denotes here the first eigenvalue of a membrane stretched over D and fixed along the boundary of D. Therefore, $\lambda = \lambda_1$, if λ_1 is taken in the meaning given to it in several preceding sections; see III, 2.5).

The values of these functionals corresponding to another domain D_* will be denoted by A_*, I_*, P_*, K_*, k_*, and λ_*, respectively.

We shall assume that D_* has a certain symmetry, and we shall distinguish two cases in this respect.

If the bisector of the coordinate axes, with equation $x = y$, is a line of symmetry for D_*, we say that D_* has *simple symmetry*.

If there is an angle $2\pi/n$, where n is an integer ≥ 3, rotated through which D_* coincides with itself, we say that D_* has higher symmetry.

Let the points (x,y) and (x_*,y_*) describe the domains D and D_*, respectively. If these points are connected by the relations

$$x = ax_* \quad , \quad y = bx_* \tag{1.1}$$

where a and b are positive constants, we say that D_* is transformed into D by pure dilatation. If the points are connected by the relations

$$x = ax_* + by_* \quad , \quad y = cx_* + dy_* \tag{1.2}$$

where the determinant ad - bc of the constants a. b, c, d is positive, we say that D_* is transformed into D by affinity.

Now, we have completed the necessary preparations and can state our theorem.

We consider two cases:

(1) D_* has simple symmetry and is transformed into D by pure dilatation;

(2) D_* has higher symmetry and is transformed into D by any affinity.

For both cases we have the inequalities

$$\frac{I_*A_*^{-2}}{IA^{-2}} \leq \frac{PA^{-2}}{P_*A_*^{-2}} \leq \frac{IA^{-2}}{I_*A_*^{-2}} \tag{1.3}$$

$$\frac{I_*A_*^{-2}}{IA^{-2}} \leq \frac{K^2A^{-1}}{K_*^2A_*^{-1}} \leq \frac{IA^{-2}}{I_*A_*^{-2}} \tag{1.4}$$

$$\frac{I_*A_*^{-2}}{IA^{-2}} \leq \frac{k_*}{k} \leq \frac{IA^{-2}}{I_*A_*^{-2}} \tag{1.5}$$

$$\frac{\lambda A}{\lambda_*A_*} \leq \frac{IA^{-2}}{I_*A_*^{-2}} \tag{1.6}$$

The elementary fact that $I_*A_*^{-2} \leq IA^{-2}$ enters into all the preceding inequalities. We can rephrase the theorem by saying that there are two cases in which all the seven functionals

$P^{-1}A^4I^{-1}$, PI^{-1}

K^2AI^{-1} , $K^{-2}A^3I^{-1}$

kA^2I^{-1} , $k^{-1}A^2I^{-1}$

λA^3I^{-1}

attain their maximum when D becomes D_*: first, when D_* has simple symmetry and D varies through all the transforms of D_* by pure dilatation, and second, when D_* has higher symmetry, and D varies through all the transforms of D_* by affinity. In comparison with the first case, the symmetry is more refined and the class of admissible domains is wider in the second case. The theorem yields both upper and lower estimates for the ratios P/P_*, K/K_*, k/k_*, and an upper estimate for λ/λ_*. We shall see that, in fact, these estimates depend only on the transformation that changes D_* into D, but are otherwise independent on the shape and size of these domains.

1.2. We consider a given domain D in space referred to the rectangular coordinates x, y, z and the functionals K, V, J, and H of D.

K is the electrostatic capacity of D, or rather of the body B whose exterior is D.

V is the volume of D.

(1.7) $$J = \iiint_D (x^2 + y^2 + z^2)\,dx\,dy\,dz$$

provided that the center of gravity of D is taken as the origin of the coordinate system; J is the radial moment of inertia of D.

(1.8) $$H = \iint \frac{dS}{h}$$

[cf. 15, p. 68-69]. The double integral is extended over the whole surface S of D an element of which is denoted by dS, and h is the distance of the origin from the tangent plant at dS. We assume here that the origin is at the

center of gravity of D and that D is convex. (To assume that D is star-shaped with respect to the origin would also be sufficient.)

The quantities K_*, V_*, J_*, and H_* are connected with the domain D_* in the same way as K, V, J, and H are with D.

We assume that D_* is symmetrical and distinguish two cases.

If the three bisecting planes of the first octant of the coordinate system (with equations $y = z$, $z = x$, and $x = y$, respectively) are planes of symmetry for D_*, we say that D_* has <u>simple symmetry</u>.

If D has the rotational symmetry of any of the five regular solids (and, possibly, some planes of symmetry in addition) we say that D_* has <u>higher symmetry</u>. (Therefore, D_* has higher symmetry if, and only if, it is brought to coincidence with itself by all the operations of one of the following seven finite groups: T, O, I, T_i, O_i, I_i, T[O; for the notation, see [8].)

Let the points (x,y,z) and (x_*,y_*,z_*) describe the domains D and D_*, respectively. If these points are connected by the relations

(1.9) $$x = ax_* \quad , \quad y = by_* \quad , \quad z = cz_*$$

where a, b, and c are positive constants, we say that D_* is transformed into D by <u>pure dilatation</u>. If the points are connected by the relations

(1.10) $$\begin{aligned} x &= a_{11}x_* + a_{12}y_* + a_{13}z_* \\ y &= a_{21}x_* + a_{22}y_* + a_{23}z_* \\ z &= a_{31}x_* + a_{32}y_* + a_{33}z_* \end{aligned}$$

and the determinant of the nine constants $a_{11}, a_{12}, \ldots, a_{33}$ is positive, we say that D_* is transformed into D by <u>affinity</u>.

<u>We consider two cases</u>:

(1) <u>D_* has simple symmetry and is transformed into D by pure dilatation</u>;

(2) <u>D_* has higher symmetry and is transformed into D by any affinity</u>.

In both cases

$$(1.11) \qquad \frac{J_* V_*^{-2}}{J V^{-2}} \leq \frac{K}{K_*} \leq \frac{H}{H_*} \quad .$$

As in the theorem of the foregoing Section 1.1, both sides of this inequality depend, in fact, only on the transformation. In the first case, the planes of symmetry and the directions of dilatation are in a definite geometric relation to each other, yet no such relation is necessary in the second case; this also is analogous to the situation in the plane.

1.3. We shall drop the somewhat cumbrous notation x_*, y_*, z_* and we shall instead use ξ, η, ζ instead, respectively. Sometimes, when convenient, we shall write x_1, x_2, x_3, ξ_1, ξ_2, and ξ_3 for x, y, z, ξ, η, and ζ, respectively. For example, the relation (1.10) will appear in matrix form as

$$(1.12) \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \quad .$$

We shall need sometimes the inverse matrix with elements α_{ik} and use, instead of (1.12), the equivalent relation

$$(1.13) \qquad \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad .$$

Instead of (1.12) and (1.13) we can also write, more concisely,

$$(1.14) \qquad x_i = \sum a_{ik} \xi_k \quad , \quad \xi_i = \sum \alpha_{ik} x_k \quad .$$

The first partial derivatives of the function f(x,y.z) with respect to its first, second, and third argument, usually denoted by f_x, f_y, f_z, will be sometimes denoted by $f_1(x,y,z)$, $f_2(x,y.z)$, $f_3(x,y,z)$, respectively, and similar notation will be occasionally used for functions of two variables.

We lay down a convention for the present chapter: The domain of integration is D, taken with 2 or 3 dimensions, if the element of integration is $dx\,dy$ or $dx\,dy\,dz$, yet it is D_*, if the element of integration is $d\xi\, d\eta$ or $d\xi\, d\eta\, d\zeta$. To surface integrals, as considered in Section 2.5, the convention does not apply, of course.

2. Symmetric Fields

2.1. We say that the function $\varphi(\xi, \eta, \zeta)$ defined in the three dimensional domain D_* admits the symmetry of D_* if

(2.1) $$\varphi(\xi_1', \xi_2', \xi_3') = \varphi(\xi_1, \xi_2, \xi_3)$$

for an arbitrary point (ξ_1, ξ_2, ξ_3) of D_* provided that the orthogonal transformation (rotation, or rotatory reflection)

$$\xi_i' = b_{i1}\xi_1 + b_{i2}\xi_2 + b_{i3}\xi_3 \qquad (i = 1,2,3)$$

transforms D_* into itself. We say that the vector field $\kappa_i(\xi_1, \xi_2, \xi_3)$ $(i = 1,2,3)$ defined in D_* admits the symmetry of D_* if, for an arbitrary point (ξ_1, ξ_2, ξ_3) of D_*, for the same group of transformations, and for $i = 1,2,3$,

(2.2) $$\kappa_i(\xi_1', \xi_2', \xi_3') = b_{i1}\kappa_1(\xi_1, \xi_2, \xi_3) + b_{i2}\kappa_2(\xi_1, \xi_2, \xi_3) + b_{i3}\kappa_3(\xi_1, \xi_2, \xi_3) \quad .$$

For example, if the function φ admits the symmetry of D_*, also the field of its gradient φ_ξ, φ_η, φ_ζ admits that symmetry. Similar definitions hold for scalar and vector fields in a plane. We wish to prove the following:

(a) The function $\varphi(\xi, \eta)$ and the vector with components $\kappa_1(\xi, \eta)$, $\kappa_2(\xi, \eta)$ are defined in, and admit the symmetry of, the two dimensional domain D_*. If D_* has simple symmetry, then

(2.3) $$\iint [\varphi_\xi(\xi, \eta)]^2 d\xi\, d\eta = \iint [\varphi_\eta(\xi, \eta)]^2 d\xi\, d\eta \quad ,$$

(2.4) $$\iint [\kappa_1(\xi, \eta)]^2 d\xi\, d\eta = \iint [\kappa_2(\xi, \eta)]^2 d\xi\, d\eta \quad .$$

... D_* has higher symmetry, we have in addition

(2.5) $$\iiint \varphi_\xi(\xi,\eta)\varphi_\eta(\xi,\eta)d\xi d\eta = 0 \quad ,$$

(2.6) $$\iiint \kappa_1(\xi,\eta)\kappa_2(\xi,\eta)d\xi d\eta = 0 \quad .$$

(b) <u>The function</u> $\varphi(\xi,\eta,\zeta)$ <u>and the vector with components</u> $\kappa_1(\xi,\eta,\zeta)$, $\kappa_2(\xi,\eta,\zeta)$, $\kappa_3(\xi,\eta,\zeta)$ <u>are defined in, and admit the symmetry of. the three dimensional</u> domain D_*. <u>If</u> D_* <u>has simple symmetry, then</u>

(2.7) $$\iiint \varphi_\xi^2 d\xi d\eta d\zeta = \iiint \varphi_\eta^2 d\xi d\eta d\zeta = \iiint \varphi_\zeta^2 d\xi d\eta d\zeta \quad ,$$

(2.8) $$\iiint \kappa_1^2 d\xi d\eta d\zeta = \iiint \kappa_2^2 d\xi d\eta d\zeta = \iiint \kappa_3^2 d\xi d\eta d\zeta \quad .$$

<u>If, however,</u> D_* <u>has higher symmetry, we have in addition</u>

(2.9) $$\iiint \varphi_\eta\varphi_\zeta d\xi d\eta d\zeta = \iiint \varphi_\zeta\varphi_\xi d\xi d\eta d\zeta = \iiint \varphi_\xi\varphi_\eta d\xi d\eta d\zeta = 0 \ ,$$

(2.10) $$\iiint \kappa_2\kappa_3 d\xi d\eta d\zeta = \iiint \kappa_3\kappa_1 d\xi d\eta d\zeta = \iiint \kappa_1\kappa_2 d\xi d\eta d\zeta = 0 \ .$$

In these statements, for all integrals considered, the domain of integration is D_*, which has to be regarded as two- or three-dimensional, as the context requires, and the same notation is adopted till the end of Section 2.4, in accordance with what has been said at the end of Section 1.3.

If the function $\varphi(\xi,\eta,\zeta)$ has the symmetry of D_*, also the field of its gradient has the symmetry of D_*. obviously. Therefore, of the eight relations asserted by (a) and (b) it will be enough to prove four. For instance, (2.9) is a particular case of (2.10).

<u>2.2</u>. Insofar as they deal with simple symmetry, the statements (a) and (b) of the foregoing Section 2.1 are obvious. Therefore, we begin by discussing the case of higher symmetry in two dimensions.

We assume that the domain D_* is transformed into itself by the rotation

(2.11) $$\xi' = \xi\cos\theta - \eta\sin\theta \quad , \quad \eta' = \xi\sin\theta + \eta\cos\theta$$

where

$$(2.12) \qquad 0 < \theta = 2\pi/n < \pi$$

and that

$$(2.13) \qquad \begin{aligned} K_1(\xi',\eta') &= K_1(\xi,\eta)\cos\theta - K_2(\xi,\eta)\sin\theta \quad , \\ K_2(\xi',\eta') &= K_1(\xi,\eta)\sin\theta + K_2(\xi,\eta)\cos\theta \end{aligned}$$

in all points (ξ,η) of D_*; of course, ξ', η' in (2.13) are expressed by (2.11). We introduce the abbreviations

$$(2.14) \qquad \iint K_1^2 d\xi\, d\eta = P \ , \ \iint K_1 K_2 d\xi\, d\eta = Q \quad , \quad \iint K_2^2 d\xi\, d\eta = R$$

and the indeterminates x, y; we assume that x', y' are so expressed in terms of x, y as ξ', η' are in terms of ξ, η, by (2.11). Then, obviously

$$(2.15) \qquad \iint [x' K_1(\xi',\eta') + y' K_2(\xi',\eta')]^2 d\xi' d\eta' = \iint [x K_1(\xi,\eta) + y K_2(\xi,\eta)]^2 d\xi\, d\eta \quad .$$

Introduce ξ, η as new variables of integration of the left-hand side, substitute for x' and y' their expressions in terms of x and y, compare the coefficients of x^2 and xy on both sides of (2.15) and use (2.14). We obtain so

$$\begin{aligned} (P-R)\sin^2\theta - 2Q\cos\theta\sin\theta &= 0 \quad , \\ (P-R)\cos\theta\sin\theta + 2Q\sin^2\theta &= 0 \quad , \end{aligned}$$

that is, a system of two homogeneous linear equations for the unknowns P-R and 2Q with the determinant $\sin^2\theta$ which is positive by virtue of (2.12). Therefore, P-R and 2Q vanish; this is (see (2.14)) precisely the desired conclusion (2.4) (2.6), which contains (2.3), (2.5) as a particular case. The theorem (a) of Section 2.1 is proved.

2.3. The proof for (b) of Section 2.1 is similar. Let the vector field $K_1(\xi,\eta,\zeta)$, $K_2(\xi,\eta,\zeta)$, $K_3(\xi,\eta,\zeta)$ have the symmetry of D_* and introduce the indeterminates x, y, z. The quadratic form in x, y, z

$$(2.16) \qquad \iiint [K_1(\xi,\eta,\zeta)x + K_2(\xi,\eta,\zeta)y + K_3(\xi,\eta,\zeta)z]^2 d\xi\, d\eta\, d\zeta$$

obviously admits all the orthogonal substitutions of the group that brings D_* in coincidence with itself. In other words, (2.16) is a quadratic invariant of the group. It is obvious that $x^2 + y^2 + z^2$ is an invariant of any orthogonal group. It is not so obvious, but it can be shown [8] that the seven groups listed in Section 2.1 have no quadratic invariant linearly independent of $x^2 + y^2 + z^2$. (This property characterizes the seven groups.) Therefore, the quadratic form (2.16) has, in fact, the shape

$$\text{const.}\ (x^2 + y^2 + z^2) \quad .$$

Comparing coefficients, we obtain immediately the desired conclusion (2.8), (2.10) from which (2.7), (2.9) follow.

2.4. If the center of gravity of the two dimensional domain D_* is at the origin, the function $(\xi^2 + \eta^2)/2$ and the field of the vector ξ, η (its gradient) obviously admit the symmetry of D_*. We infer from (a) that, if D_* has simple symmetry

$$\iint \xi^2 d\xi\, d\eta = \iint \eta^2 d\xi\, d\eta \quad . \tag{2.17}$$

If, however, D_* has higher symmetry, we have in addition

$$\iint \xi\eta\, d\xi\, d\eta = 0 \quad . \tag{2.18}$$

Similarly, if the center of gravity of the three dimensional domain D_* is at the origin, and D_* has simple symmetry

$$\iiint \xi^2 d\xi\, d\eta\, d\zeta = \iiint \eta^2 d\xi\, d\eta\, d\zeta = \iiint \zeta^2 d\xi\, d\eta\, d\zeta \quad . \tag{2.19}$$

If, however, D_* has higher symmetry, we have in addition

$$\iiint \eta\zeta\, d\xi\, d\eta\, d\zeta = \iiint \zeta\xi\, d\xi\, d\eta\, d\zeta = \iiint \xi\eta\, d\xi\, d\eta\, d\zeta = 0 \quad . \tag{2.20}$$

The relations (2.19) and (2.20) express the fact that the ellipsoid of inertia of D_* around its center of gravity reduces to a sphere. We inferred this from the circumstance that D_* admits one of the seven finite groups listed in Section 2.1. If D_* admitted any finite group different from these seven, such an inference would not be valid.

2.5. We can use the method of Section 2.2 and 2.3 in dealing with H_*. In fact, we may assume that the boundary of the three-dimensional domain D_* is given by the equation

(2.21) $$\Phi(\xi, \eta, \zeta) = 1$$

where $\Phi(\xi, \eta, \zeta)$ is a continuously differentiable homogeneous function of degree 1. We have, therefore,

(2.22) $$\xi\Phi_\xi + \eta\Phi_\eta + \zeta\Phi_\zeta = 1$$

and so the equation of a tangent plane is

(2.23) $$x\Phi_\xi + y\Phi_\eta + z\Phi_\zeta = 1 \quad .$$

Let $\cos\alpha$, $\cos\beta$, $\cos\gamma$ stand for the direction cosines of the normal at the point ξ, η, ζ. We take h to have the meaning defined in Section 2.1, and we obtain from (2.23) that

(2.24) $$\frac{\cos\alpha}{h} = \Phi_\xi \quad , \quad \frac{\cos\beta}{h} = \Phi_\eta \quad , \quad \frac{\cos\gamma}{h} = \Phi_\zeta \quad .$$

It follows from (1.8) and (2.24) that

(2.25) $$H_* = \iint(\Phi_\xi \cos\alpha + \Phi_\eta \cos\beta + \Phi_\zeta \cos\gamma)dS \quad .$$

This double integral, as all those that follow in the present section, is extended over the whole surface of D_*. Consider now the quadratic form in the indeterminates x, y, z

(2.26) $$\iint(x\Phi_\xi + y\Phi_\eta + z\Phi_\zeta)(x\cos\alpha + y\cos\beta + z\cos\gamma)dS \quad .$$

It obviously admits the symmetry of D_*. Therefore, if D_* has higher symmetry, the quadratic form (2.26) must be a constant multiple of the form $x^2 + y^2 + z^2$ and so

(2.27) $$\iint\Phi_\xi \cos\alpha\, dS = \iint\Phi_\eta \cos\beta\, dS = \iint\Phi_\zeta \cos\gamma\, dS$$

(2.28) $$\begin{aligned}\iint(\Phi_\eta \cos\gamma + \Phi_\zeta \cos\beta)dS &= \iint(\Phi_\zeta \cos\alpha + \Phi_\xi \cos\gamma)dS \\ &= \iint(\Phi_\xi \cos\beta + \Phi_\eta \cos\alpha)dS = 0 \quad .\end{aligned}$$

Obviously, (2.27) (but not (2.28)) remains valid under the different assumption that D_* has simple symmetry.

We shall need later another form of the integral (2.25):

$$H_* = \iint (\Phi_\xi \, d\eta \, d\zeta + \Phi_\eta \, d\zeta \, d\xi + \Phi_\zeta \, d\xi \, d\eta) \qquad (2.29)$$

3. Affine Transplantation of Symmetric Fields.

3.1. The point (x,y,z) (or x_1,x_2,x_3) describes the domain D, and the point (ξ,η,ζ) (or (ξ_1,ξ_2,ξ_3)) describes D_*. These points are connected by (1.12) (or (1.13), or (1.14)), and so the domains D and D_* themselves are connected by the affinity (1.12).

A function $\varphi(\xi,\eta,\zeta)$ defined in D_* determines a scalar field in D_*, and three functions $\kappa_i(\xi,\eta,\zeta)$, $i = 1,2,3$, a vector field. We transplant these fields by affinity into D, defining there $f(x,y,z)$ and $q_i(x,y,z)$ by the equations

$$f(x,y,z) = \varphi(\xi,\eta,\zeta) \quad , \qquad (3.1)$$

$$q_i(x,y,z) = a_{i1}\kappa_1(\xi,\eta,\zeta) + a_{i2}\kappa_2(\xi,\eta,\zeta) + a_{i3}\kappa_3(\xi,\eta,\zeta) \quad , \qquad (3.2)$$

$i = 1,2,3$. Equations (3.1) and (3.2) are valid at all points of D_*; of course, x, y, z are expressed in terms of (ξ,η,ζ) according to (1.12) and so $f(x,y,z)$ and $q_i(x,y,z)$ are defined in all points (x,y,z) of D.

Taking the partial derivative of (3.1) with respect to x_i, we obtain from (1.13) that

$$f_i(x,y,z) = \varphi_1(\xi,\eta,\zeta)\alpha_{1i} + \varphi_2(\xi,\eta,\zeta)\alpha_{2i} + \varphi_3(\xi,\eta,\zeta)\alpha_{3i} \quad , \qquad (3.3)$$

$i = 1,2,3$. Comparing (3.2) and (3.3), we see that to take the gradient of a transplanted scalar field is not necessarily the same as to transplant the original gradient field. (The two coincide, however, when the a_{ik} form an orthogonal matrix, which was the case considered in Section 2.1.)

When D and D_*, described respectively by (x,y) and (ξ,η), are two-dimensional, the connecting affinity takes the shape (1.2), (with ξ,η for

x_*, y_*) and the two-dimensional fields and their transplantation are correspondingly defined.

We shall study the transplantation of scalar and vector fields by assuming more and more symmetry of the original fields as we progress.

3.2. At first, we assume no symmetry of D_*. Still we can assert, when D and D_* are two-dimensional, that

(3.4) $$A = (ad - bc)A_* \quad ,$$

(3.5) $$q_{1x} + q_{2y} = \kappa_{1\xi} + \kappa_{2\eta} \quad ,$$

(3.6) $$\int (q_1 dy - q_2 dx) = (ad - bc)\int (\kappa_1 d\eta - \kappa_2 d\xi) \quad ;$$

the integrals in (3.6) are extended along corresponding closed curves in D and D_*, respectively.

When D and D_* are three-dimensional, we have

(3.7) $$V = |a_{ik}| V_* \quad ,$$

(3.8) $$q_{1x} + q_{2y} + q_{3z} = \kappa_{1\xi} + \kappa_{2\eta} + \kappa_{3\zeta} \quad ,$$

(3.9) $$\iint (q_1 dydz + q_2 dzdx + q_3 dxdy) = |a_{ik}| \iint (\kappa_1 d\eta d\zeta + \kappa_2 d\zeta d\xi + \kappa_3 d\xi d\eta) \quad ;$$

the integrals in (3.9) are extended over corresponding closed surfaces in D and D_*, respectively.

We can express (3.8) by saying that the divergence of a vector field is an affine invariant. Of course, (3.8) contains (3.5) as a special case and suggests strongly (3.9) (but does not prove it immediately, since the interior of a closed surface that lies in the field need not wholly belong to the field.) As the proofs for (3.4), (3.5), (3.7), and (3.8) consist in straightforward calculations, we restrict ourselves to a few remarks on the proof of (3.9).

We let the parameters p and q determine the position of a variable point on the surface lying in D. Then, on the left-hand side of (3.9)

(3.10)
$$dy\,dz = \begin{vmatrix} y_p & y_q \\ z_p & z_q \end{vmatrix} dp\,dq$$

and, by (1.12),

(3.11)
$$\begin{vmatrix} y_p & y_q \\ z_p & z_q \end{vmatrix} = \begin{vmatrix} a_{21}\xi_p + a_{22}\eta_p + a_{23}\zeta_p & a_{21}\xi_q + a_{22}\eta_q + a_{23}\zeta_q \\ a_{31}\xi_p + a_{32}\eta_p + a_{33}\zeta_p & a_{31}\xi_q + a_{32}\eta_q + a_{33}\zeta_q \end{vmatrix}$$

$$= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}\begin{vmatrix} \eta_p & \eta_q \\ \zeta_p & \zeta_q \end{vmatrix} + \begin{vmatrix} a_{23} & a_{21} \\ a_{33} & a_{31} \end{vmatrix}\begin{vmatrix} \zeta_p & \zeta_q \\ \xi_p & \xi_q \end{vmatrix} + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}\begin{vmatrix} \xi_p & \xi_q \\ \eta_p & \eta_q \end{vmatrix} .$$

Since the α_{ik} are lements of the matrix inverse to that of the a_{ik}, we may express them as cofactors of the latter, and may therefore write (3.11), with the notation (3.10) as the first of the following three equations

(3.12)
$$\begin{aligned} dy\,dz &= |a_{ik}|(\alpha_{11}d\eta d\zeta + \alpha_{21}d\zeta d\xi + \alpha_{31}d\xi d\eta) , \\ dz\,dx &= |a_{ik}|(\alpha_{12}d\eta d\zeta + \alpha_{22}d\zeta d\xi + \alpha_{32}d\xi d\eta) , \\ dx\,dy &= |a_{ik}|(\alpha_{13}d\eta d\zeta + \alpha_{23}d\zeta d\xi + \alpha_{33}d\xi d\eta) . \end{aligned}$$

Substituting for $q_1, \ldots,$ $dy\,dz, \ldots$ their values from (3.2) and (3.12), respectively, on the left-hand side of (3.9), we obtain the right-hand side by observing the relations between the elements of inverse matrices.

<u>3.3</u>. We assume now that D_* has <u>simple symmetry</u>, but we assume also that the affinity connecting D with D_* is a <u>pure dilatation</u>, given by (1.1) or by (1.9) (with ξ, η, ζ for x_*, y_*, z_*) according as D and D_* are two- or three-dimensional.

If D and D_* are two-dimensional

(3.13)
$$I = \frac{1}{2}\, ab(a^2 + b^2) I_* .$$

(3.14)
$$\iint (f_x^2 + f_y^2)\,dx\,dy = \frac{ab}{2}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)\iint(\varphi_\xi^2 + \varphi_\eta^2)\,d\xi\,d\eta$$

(3.15)
$$\iint (q_1^2 + q_2^2)\,dx\,dy = \frac{1}{2}\, ab(a^2 + b^2)\iint(K_1^2 + K_2^2)\,d\xi\,d\eta .$$

If D and D_* are three-dimensional

(3.16) $$J = \frac{1}{3}\, abc(a^2 + b^2 + c^2) J_* \quad ,$$

(3.17) $$H = \frac{abc}{3}\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) H_* \quad ,$$

(3.18) $$\iiint (f_x^2 + f_y^2 + f_z^2)\,dx\,dy\,dz = \frac{abc}{3}\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)\iiint (\varphi_\xi^2 + \varphi_\eta^2 + \varphi_\zeta^2)\,d\xi\,d\eta\,d\zeta \, ,$$

(3.19) $$\iiint (q_1^2 + q_2^2 + q_3^2)\,dx\,dy\,dz = \frac{1}{3}\, abc(a^2 + b^2 + c^2)\iiint (\kappa_1^2 + \kappa_2^2 + \kappa_3^2)\,d\xi\,d\eta\,d\zeta \quad .$$

The proofs for (3.13), (3.14), (3.15), (3.16), (3.17), (3.18), and (3.19) are based on (2.17), (2.3), (2.4), (2.19), (2.27), (2.7), and (2.8), respectively. These proofs are so straightforward that we can leave them to the reader, who may find it advantageous to use (2.29) when he deals with H.

3.4. Finally, we assume that D_* has higher symmetry and we do not restrict the affinity that changes D_* into D.

If D and D_* are two-dimensional

(3.20) $$I = \frac{1}{2}\,(ad - bc)(a^2 + b^2 + c^2 + d^2) I_* \quad ,$$

(3.21) $$\iint (f_x^2 + f_y^2)\,dx\,dy = \frac{a^2 + b^2 + c^2 + d^2}{2(ad - bc)}\iint (\varphi_\xi^2 + \varphi_\eta^2)\,d\xi\,d\eta \quad ,$$

(3.22) $$\iint (q_1^2 + q_2^2)\,dx\,dy = \frac{1}{2}\,(ad - bc)(a^2 + b^2 + c^2 + d^2)\iint (\kappa_1^2 + \kappa_2^2)\,d\xi\,d\eta \quad .$$

If D and D_* are three-dimensional

(3.23) $$J = \frac{1}{3}\,|a_{ik}| \sum\sum a_{ik}^2\, J_*$$

(3.24) $$H = \frac{1}{3}\,|a_{ik}| \sum\sum \alpha_{ik}^2\, H_*$$

(3.25) $$\iiint (f_x^2 + f_y^2 + f_z^2)\,dx\,dy\,dz = \frac{1}{3}|a_{ik}| \sum\sum \alpha_{ik}^2 \iiint (\varphi_\xi^2 + \varphi_\eta^2 + \varphi_\zeta^2)\,d\xi\,d\eta\,d\zeta$$

(3.26) $$\iiint (q_1^2 + q_2^2 + q_3^2)\,dx\,dy\,dz = \frac{1}{3}|a_{ik}| \sum\sum a_{ik}^2 \iiint (\kappa_1^2 + \kappa_2^2 + \kappa_3^2)\,d\xi\,d\eta\,d\zeta \quad .$$

For example, the proof of (3.25) results immediately from (3.3), (2.7), and (2.8), and the proofs of the other relations are likewise straightforward, except perhaps the proof for (3.24). In order to derive

(3.24) let $F(x,y,z)$ be the homogeneous function of degree 1 that arises in the equation

(3.27) $$F(x,y,z) = 1$$

of the surface of D, which corresponds to (2.21), the equation of the surface of D^*. Obviously

(3.28) $$F(x,y,z) = \Phi(\xi,\eta,\zeta)$$

for any point (ξ,η,ζ) of space, provided that x, y, and z are expressed in terms of ξ, η, and ζ according to (1.12). Then

(3.29) $$F_i(x,y,z) = \Phi_1(\xi,\eta,\zeta)\alpha_{1i} + \Phi_2(\xi,\eta,\zeta)\alpha_{2i} + \Phi_3(\xi,\eta,\zeta)\alpha_{3i}$$

for $i = 1,2,3$ and, corresponding to (2.29),

(3.30) $$H = \iint (F_x\,dy\,dz + F_y\,dz\,dx + F_z\,dx\,dy) \quad .$$

From (3.30), (3.29), (3.12), (2.27), (2.28), and (2.29), we easily obtain (3.24).

The transformation formulas of this section go over into those of the foregoing Section 3.3 if the affinity reduces to a pure dilatation. The difference between the important formulas (3.25) and (3.26) corresponds to the difference between (3.24) and (3.23) and is due to the difference between (3.3) and (3.2).

The Dirichlet integral, which is a conformal invariant, is not an affine invariant. The simplicity of the formulas (3.18) and (3.25) is due to the underlying symmetry assumptions concerning D_* and $\varphi(\xi,\eta,\zeta)$.

4. Applications.

4.1. We apply now the preceding general results to the various functionals enumerated in Section 1.1. We start with the torsional rigidity P which was treated in the general case in Section 1.1 of Chapter I. It was found convenient to plot $tP(t)^{-1}$ against the variable $\tau = t^{-2}$ if the plane domain D was subjected to a one-sided stretching with the parameter t. If $f(x,y,z)$ is

the stress function of D, we expressed the slope of the above curve at the abscissa τ by (I. 1, 8) as

$$a = \tau^{-3/2} P^{-2} \iint_D f_x^2 \, dx \, dy \quad . \tag{4.1}$$

Let us suppose now that D_* is of simple or higher symmetry; since the stress function of a domain is uniquely determined and since D_* goes into itself under a symmetry transformation, the stress function $\varphi(\xi,\eta,\zeta)$ of D_* must be invariant under the symmetry transformations. Thus, (2.3) must hold in this case, and, by (I, 1.2), we have:

$$P_* = 2\iint \varphi_\xi^2 \, d\xi \, d\eta \quad . \tag{4.2}$$

Thus, in this special case, the slope a can be expressed in the form

$$a = \frac{1}{2P^*} \tau^{-3/2} = \frac{1}{\tau} \frac{1}{2} \tau^{-1/2} P_*^{-1} \quad . \tag{4.3}$$

Since we plotted the ordinate $\tau^{-1/2} P^{-1}$ against the abscissa τ, we may characterize the slope of this curve at the point τ belonging to the symmetric domain D^* as follows. The tangent at the point belonging to D^* intercepts the vertical axis at one half of the ordinate of this curve point. This fact leads to an easy construction of this tangent if this curve point is known. Since we know already that the curve is convex everywhere the knowledge of such a point leads to a very useful piece of information on the location of the whole curve. This fact will be utilized in the Appendix in order to obtain numerical estimates for the torsional rigidity of triangles.

The symmetry of a domain can be applied in the same way in various other problems in order to determine the slope of curves for given functionals under stretching of their domain of definition. Consider, for example, the electrostatic capacity of a symmetric domain D_* in space. By reason of symmetry, we have the equations (2.7) for the conductor potential $\varphi(\xi,\eta,\zeta)$ of D_* since this uniquely determined domain function must admit the symmetry of D_*. Hence:

$$(4.4) \qquad K_* = \frac{1}{4\pi}\iiint(\varphi_\xi^2 + \varphi_\eta^2 + \varphi_\zeta^2)\,d\xi\,d\eta\,d\zeta = \frac{3}{4\pi}\iiint\varphi_\xi^2\,d\xi\,d\eta\,d\zeta \quad .$$

On the other hand, if we plot the expression $t^{-1}K(t)$ against $\tau = t^{-2}$ in the case of one-sided stretching, we find that the slope of the curve is given by (I. 4.6) which leads now to the result:

$$(4.5) \qquad a = \frac{1}{3}\,t K_* = \frac{1}{\tau}\cdot\frac{1}{3}\,(t^{-1}K_*) \quad .$$

This can again be interpreted geometrically: The tangent to the curve at the point corresponding to D_* intercepts the vertical axis at 2/3 of the ordinate.

Finally, let D_* be a plane symmetric domain and $\varphi(\xi,\eta)$ denote the membrane eigenfunction belonging to the lowest eigenvalue λ_*. Since the lowest eigenvalue is non-degenerate the eigenfunction is uniquely determined and has necessarily the symmetry of the domain. Thus, by (2.3),

$$(4.6) \qquad \lambda_* = \iint(\varphi_\xi^2 + \varphi_\eta^2)\,d\xi\,d\eta = 2\iint\varphi_\xi^2\,d\xi\,d\eta \quad .$$

We showed, on the other hand, in Chapter III that if we plot $\lambda(t)$ against $\tau = t^{-2}$ in the case of one-sided stretching, we obtain a curve with the slope

$$(4.7) \qquad a = \tau^{-1}\iint_{D(t)} u_x^2\,dx\,dy \quad .$$

Thus, if τ^* is just the abscissa of D_*, we have by (4.6)

$$(4.8) \qquad a = \frac{1}{\tau^*}\;\frac{1}{2}\lambda_* \quad .$$

Again, the tangent at the point (τ^*, λ^*) has the intercept $\frac{1}{2}\lambda_*$ on the vertical axis.

It should be observed that no similar conclusion is valid for the case of the eigenvalue μ_2 of the free boundary value problem (III. 2.14). This eigenvalue may be degenerate and admit more than one eigenfunction. Hence, it is not sure that the eigenfunctions of D_* will have the symmetry of the domain and formula (2.3) cannot be used.

4.2. We consider next a domain D_* in space with simple symmetry. Let $\varphi(\xi,\eta,\zeta)$ denote its conductor potential, K_* is capacity and $\underline{K} = \nabla\varphi$ the corresponding gradient vector field. Let D be obtained from D_* by a pure dilatation (1.9). We define then a test function

$$(4.9) \qquad f(x,y,z) = \varphi\left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right)$$

which is well-defined in D, has the boundary value 1 and is admissible in the Dirichlet inequality (I, 4.4). Thus, we have

$$(4.10) \qquad K \leq \frac{1}{4\pi} \iiint (f_x^2 + f_y^2 + f_z^2)\,dx\,dy\,dz \quad .$$

Using now (3.18) which holds because of the symmetry of φ, we obtain

$$(4.11) \qquad K \leq \frac{abc}{3}\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)K_* \quad .$$

Another estimate for K may be deduced from Thomson's principle (I. 4.38). We transplant the vector field K_1 by (3.2) into the domain D and use it there as a test field in the Thomson inequality which we formulate as follows (see (I. 4.35)):

$$(4.12) \qquad \frac{1}{4\pi K} \leq \frac{\iiint q^2\,dx\,dy\,dz}{\left(\iint q\cdot n\,d\sigma\right)^2} \quad .$$

Hence, we find by (3.9) and (3.19)

$$(4.13) \qquad \frac{1}{K} \leq \frac{4\pi \iiint (q_1^2 + q_2^2 + q_3^2)\,dx\,dy\,dz}{\left[\iint (q_1\,dy\,dz + q_2\,dx\,dz + q_3\,dx\,dy\right]^2} = \frac{a^2 + b^2 + c^2}{3abc}\,\frac{1}{K_*} \quad .$$

Thus, we have:

$$(4.14) \qquad \frac{3abc}{a^2 + b^2 + c^2} \leq \frac{K}{K_*} \leq \frac{abc}{3}\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) \quad .$$

This may also be formulated as

$$(4.15) \qquad \frac{3}{\frac{a^3}{aK_*} + \frac{b^3}{bK_*} + \frac{c^3}{cK_*}} \leq \frac{K}{abc} \leq \frac{1}{3}\left(\frac{aK_*}{a^3} + \frac{bK_*}{b^3} + \frac{cK_*}{c^3}\right) \quad .$$

here the lower and upper bounds for the ratio $\frac{K}{abc}$ appear as the harmonic and arithmetic mean, respectively, of the ratios $\frac{aK_*}{a^3}$, $\frac{bK_*}{b^3}$, $\frac{cK_*}{c^3}$. One calls sometimes the ratio K/V the specific capacity of a body if V denotes its volume. With this notation (4.15) may be interpreted as follows: Under a pure dilatation with parameters a, b, c the specific capacity lies between the harmonic and arithmetic means of the specific capacities which are obtained under similarities with the coefficients a, b, and c, respectively.

If the domain D_* is the exterior of a sphere of radius 1, the domain D is the exterior of an ellipsoid with the axes a, b, c and (4.15) leads to the interesting result:

The specific capacity of an ellipsoid with axes a, b, c lies between the harmonic and arithmetic means of the specific capacities of three spheres whose radii are a, b, c [14, Problem 95, p. 57].

If we compare (4.14) with the transformation laws for J and H under pure dilatation, we recognize that (4.14) implies the estimate (1.11) which was announced in Section 4.1.

In the particular case that D_* is the exterior of the unit sphere, the transformed domains are bounded by ellipsoids with axes a, b, and c. In this case, the capacity K can be expressed explicitly in terms of elliptic integrals and better estimates than (4.15) can be obtained, namely [16, p. 4, formula (1.7)]:

$$\frac{1}{3}[(bc)^{1/2} + (ca)^{1/2} + (ab)^{1/2}] \leq K \leq \frac{1}{3}[a + b + c] \quad (4.15')$$

If D_* is the exterior of the unit cube, the domains D will be the outsides of rectangular boxes with sides a, b, and c. Here symmetrization leads to the result [15, p. 159]:

$$K^*(abc)^{1/3} \leq K \quad (4.15'')$$

The result (4.15) seems new, however, for the general class of domains with higher symmetry. We mention as an example the tetrahedron defined by the inequalities

$$x + y + z \leq 1 \quad , \quad x \geq 0 \quad , \quad y \geq 0 \quad , \quad z \geq 0 \quad . \tag{4.15'''}$$

This "trirectangular isosceles tetrahedron" has the required symmetry and its exterior may serve as a domain D_*.

4.3. In the case that D_* has a higher symmetry we can extend our results to the case of general affinities. If $\varphi(\xi,\eta,\zeta)$ and $\underline{k} = \nabla\varphi$ are the correct conductor potential and gradient field of D_* and if D has been derived from D by an affinity (1.10) we transplant φ and $\underline{k}$ by means of (3.1) and (3.2). We use the new fields $f(x,y,z)$ and $q_i(x,y,z)$ as test fields in the Dirichlet and Thomson principle for the capacity K. We obtain by (3.25) and the Dirichlet principle

$$K \leq \frac{1}{4\pi} \iiint (f_x^2 + f_y^2 + f_z^2)\,dx\,dy\,dz = \frac{1}{3}\,|a_{ik}|\sum\sum \alpha_{ik}^2 K_* \quad , \tag{4.16}$$

while Thomson's principle (4.12) and (3.9), (3.26) yield

$$\frac{1}{4\pi K} \leq \frac{1}{3}\,|\alpha_{ik}|\sum\sum a_{ik}^2 \cdot \frac{1}{4\pi K_*} \quad . \tag{4.17}$$

Combining these inequalities, we arrive at

$$\frac{3|a_{ik}|}{\sum\sum a_{ik}^2} \leq \frac{K}{K_*} \leq \frac{\sum\sum \alpha_{ik}^2}{3|\alpha_{ik}|} \quad . \tag{4.18}$$

If we observe the transformations laws (3.7), (3.23), and (3.24) for the functionals V, J, and H under affinities, we see that (4.18) is equivalent to the result (1.11) announced in Section 4.1.

Inequality (4.18) contains the inequality

$$9 = \left(\sum\sum a_{ik}\,\alpha_{ik}\right)^2 \leq \sum\sum a_{ik}^2 \cdot \sum\sum \alpha_{ik}^2 \tag{4.19}$$

which is just the Cauchy inequality. In view of (3.7), (3.23), and (3.24) this obvious inequality leads to the geometric formulation

(4.20) $$H_*J_*V_*^{-2} \leq HJV^{-2}$$.

Thus, this particular functional has a minimum for the domain with highest symmetry within the class of all affinely related domains. In the case that the domain D_* is a sphere, we can make a more general statement:

Among all solids starshaped with respect to their center of gravity the sphere has the minimum value for HJV^{-2}.

In order to prove this statement we introduce the distance r from the center of gravity and the element of solid angle dw. We have then

(4.21) $$J = \frac{1}{5}\iint r^5 dw$$

(4.22) $$V = \frac{1}{3}\iint r^3 dw$$

(4.23) $$S \geq \iint r^2 dw$$.

Here all integrations are extended over the boundary of the solid and S denotes its surface area. By Hölder's inequality with $p = 1/3$, $q = 2/3$, we have

(4.24) $$3V = \iint (r^5)^{1/3}(r^2)^{2/3} dw \leq (\iint r^5 dw)^{1/3}(\iint r^2 dw)^{2/3}$$

which means by (4.21) and (4.23)

(4.25) $$(3V)^3 \leq 5JS^2$$.

On the other hand, it is known [15, p. 69, (7)] that

(4.26) $$S^2 \leq 3VH$$.

Thus, (4.25) leads to

(4.27) $$HJV^{-2} \geq \frac{9}{5} = (4\pi r)\left(\frac{4\pi r^5}{5}\right)\left(\frac{4\pi r^3}{3}\right)^{-2}$$.

This proves the asserted minimum property of the sphere and the above derivation shows also that the sphere is the only solid with this minimum property.

4.4. In a similar way we can prove the inequalities (1.3) - (1.6). We shall indicate the methods only very briefly. In the case of the torsional

rigidity P we utilize the Dirichlet principle (I, 1.5) and the Thomson principle (I, 1.18) and combine them with the methods of transplanting a scalar and vector field. If D_* has simply symmetry and D is obtained from it by a pure dilatation (1.1), we use the identities (3.5), (3.14), and (3.15) and obtain

$$(4.28) \qquad \frac{2ab}{\frac{1}{a^2}+\frac{1}{b^2}} \leq \frac{P}{P_*} \leq \frac{1}{2}ab\,(a^2+b^2) \quad .$$

If D_* has a higher symmetry and D arises through a general affinity (1.2), we apply (3.21) and (3.22) and arrive at the estimates

$$(4.29) \qquad \frac{2(ad-bc)^3}{a^2+b^2+c^2+d^2} \leq \frac{P}{P_*} \leq \frac{1}{2}\,(ad-bc)(a^2+b^2+c^2+d^2) \quad .$$

If we compare these formulas with the transformation laws (3.4) and (3.20), we recognize the validity of (1.3). We observe that all inequalities (1.3) - (1.5) contain

$$(4.30) \qquad J_*A_*^{-2} \leq JA^{-2} \quad .$$

In the case of a general affinity this is by (3.4) and (3.20) equivalent with the trivial inequality

$$(4.31) \qquad 2(ad-bc) \leq a^2+b^2+c^2+d^2 \quad .$$

It is, moreover, well known that the circle minimizes the expression JA^{-2} among all domains.

We come next to the electrostatic capacity K for which we use the Dirichlet principle (4.10) and the Thomson principle (4.12). We have to extend the linear transformations (1.1) and (1.2) into the entire (x,y,z)-space; we do this by adding the equation $z = \ell\zeta$ with arbitrary real ℓ. Suppose that D_* has a higher symmetry in the (x,y)-plane; let $\varphi(\xi,\eta,\zeta)$ be its conductor potential and $\underline{K} = \nabla\varphi$ its gradient vector field. We use the transformation

(4.32) $$x = a\xi + b\eta \quad , \quad y = c\xi + d\eta \quad , \quad z = \ell\zeta$$

and the transplantation formulas (3.1) and (3.2).

From Dirichlet's principle and (3.21), we derive easily

(4.33) $$K \leq \ell\,\frac{a^2 + b^2 + c^2 + d^2}{2(ad - bc)} \cdot \frac{1}{4\pi}\iiint [\varphi_\xi^2 + \varphi_\eta^2]\,d\xi\,d\eta\,d\zeta + \frac{1}{\ell}\,\frac{ad - bc}{4\pi}\iiint \varphi_\zeta^2\,d\xi\,d\eta\,d\zeta \quad .$$

We put this inequality into the abbreviated form

(4.34) $$K \leq \ell\,\frac{a^2 + b^2 + c^2 + d^2}{2(ad - bc)}\,P + \frac{1}{\ell}\,(ad - bc)Q$$

and observe that

(4.35) $$K_* = P + Q \quad .$$

We utilize the freedom in the choice of ℓ in order to minimize the right-hand side of our estimate and find

(4.36) $$K^2 \leq \frac{1}{2}\,(a^2 + b^2 + c^2 + d^2)4PQ \quad .$$

Since $4PQ \leq (P + Q)^2 = K_*^2$, we obtain finally

(4.37) $$\frac{K^2}{K_*^2} \leq \frac{1}{2}\,(a^2 + b^2 + c^2 + d^2)$$

which leads by virtue of (3.20) and (3.4) to the right-hand inequality of (1.4). We can derive the left-hand inequality in exactly the same way. We prove also by the same reasoning the validity of (1.4) in the case of simply symmetry.

In order to prove (1.5), we have to characterize the quantity k by a Dirichlet and a Thomson principle. It is easily seen that

(4.38) $$k = \min \frac{1}{2\pi}\iint_D (\nabla u)^2\,dx\,dy$$

where u(x,y) is any continuously differentiable function in the ring domain D which has the value 0 on one boundary curve C_o and the value 1 on the other C_1. The minimum is attained for the conductor potential U which is harmonic in D. We have by Green's formula

(4.39) $$k = -\frac{1}{2\pi}\int_{C_1} \frac{\partial U}{\partial n}\,ds = \frac{1}{2\pi}\iint_D [U_x^2 + U_y^2]\,dx\,dy \quad .$$

Consider now any continuously differentiable vector field $\underline{q}$ with

$$\nabla \cdot \underline{q} = 0 \tag{4.40}$$

and

$$\frac{1}{2\pi} \int_{C_1} \underline{q} \cdot \mathbf{n} \, ds = -1 \tag{4.41}$$

Then we have

$$\frac{1}{k} = \min \frac{1}{2\pi} \iint_D q^2 \, dx \, dy \tag{4.42}$$

and the minimum is attained for the particular vector field $\underline{q} = \frac{1}{k} \nabla U$. The proof for these statements is the same as in the case of the three-dimensional capacity and is, therefore, omitted.

The inequalities (1.5) are an immediate consequence of the extremum principles (4.38) and (4.42) and of the transformation formulas (3.13), (3.14), (3.15), (3.4), (3.20), (3.21), and (3.22).

Inequality (1.6) is obtained in the same way by use of the minimum definition of Section III, 2.1, for the lowest eigenvalue.

4.5. The symmetry of a domain D can also be used in order to obtain estimates for functionals with respect to domains D(t) which arise from D = D(1) by the partial one-sided stretching discussed in Section I, 4.3.

In order to illustrate the method, we consider the following example. We use the notation of Section I, 4.3, and assume that the body B considered is a sphere of radius R around the origin. The body B(t) will consist of a hemisphere of radius R in the half-space $x \leq 0$ and of one half of the ellipsoid with axes tR, R, and R in the half-space $x \geq 0$. Let K_o be the capacity of B and K(t) denote the capacity of B(t). If V(x,y,z) is the conductor potential in the exterior of B, we can estimate K(t) by means of (I, 4.26) with $t_o = 1$. On the other hand, we have for reasons of symmetry

$$V(x,y,z) = V(-x,y,z) \tag{4.43}$$

Consider now any continuously differentiable vector field $\underline{q}$ with

$$(4.40) \qquad \nabla \cdot \underline{q} = 0$$

and

$$(4.41) \qquad \frac{1}{2\pi} \int_{C_1} \underline{q} \cdot \mathbf{n} \, ds = -1$$

Then we have

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and the minimum is attained for the particular vector field $\underline{q} = \frac{1}{k} \nabla U$. The proof for these statements is the same as in the case of the three-dimensional capacity and is, therefore, omitted.

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$$(4.43) \qquad V(x,y,z) = V(-x,y,z) \quad .$$

Hence, if V_i is any of the partial derivatives V_x, V_y, or V_z, we have

$$(4.44) \qquad \iiint_{D^+} V_i^2 \, dx\, dy\, dz = \iiint_{D^-} V_i^2 \, dx\, dy\, dz \quad .$$

From (I, 4.27) and the equations (2.7), we conclude

$$(4.45) \qquad \iiint_{D^+} V_i^2 \, dx\, dy\, dz = \iiint_{D^-} V_i^2 \, dx\, dy\, dz = \frac{2\pi}{3} K_o \quad .$$

Hence, we derive from (I, 4.26) the estimate

$$(4.46) \qquad K(t) \leq K_o\left(\frac{1}{2} + \frac{1}{6t} + \frac{1}{3} t\right) \quad .$$

Thus, we obtained a simple estimate for the capacity of the rather complicated body B(t) in terms of the parameter of the partial stretching. Various similar results can be obtained in the same way for analogous cases of symmetry.

Chapter V

Transplantation of Harmonic Functions

1. Torsional Rigidity and the Green's Function.

1.1. In this chapter we shall discuss the dependence of various domain functionals on the parameter t in the transformation

(1.1) $$z^* = z + tf(z) \quad .$$

This transformation carries a given domain $D = D(0)$ into a sequence of domains $D(t)$; it is conformal if we assume that $f(z)$ is regular analytic in $D(0)$ and will be one-to-one for sufficiently small values of t, if we assume $f(z)$ to be regular in the closure of D.

Let $H(t)$ be the class of all functions $h(x,y)$ which are regular harmonic in $D(t)$; we shall write for sake of shortness $h = h(z)$ but observe that h does not depend analytically on the complex variable z. We can map the class $H(t)$ in a one-to-one manner into the class $H(0)$ of functions harmonic in $D(0)$ by the relation

(1.2) $$h_o(z) = h(z + tf(z)) \quad .$$

In fact, the right side function is well-defined in $D(0)$ and is harmonic there, since a conformal mapping preserves the harmonicity of functions. We may say that we have transplanted the class of all harmonic functions in $D(0)$ into the class of harmonic functions in $D(t)$ by means of the correspondence (1.2).

Various important domain functionals are closely related to the theory of harmonic functions in the domain considered and their study is facilitated by the fact that harmonic functions can be transplanted in the simple manner described.

1.2. Let $g(z, \zeta)$ be the Green's function of a simply-connected finite domain D; that is, we suppose that

a) $g(z,\zeta)$ is harmonic in the variable z, except at the point ζ

b) $g(z,\zeta) + \frac{1}{2\pi} \log|z-\zeta|$ is harmonic at ζ

c) $g(z,\zeta)$ vanishes if the variable z tends to the boundary C of D.

We consider now the function

(1.3) $$f(z) = 2\iint_D g(z,\zeta)\,d\xi\,d\eta \qquad , \quad \zeta = \xi + i\eta \quad .$$

This function is indefinitely differentiable in D and satisfies by virtue of b) the differential equation

(1.4) $$\Delta f = -2 \quad .$$

By the property c) of the Green's function we recognize that f(z) vanishes on the boundary C and hence, we may identify f(z) with the stress function of D, defined in Section I, 1.1. By means of (I, 1.4), we compute the torsional rigidity of D:

(1.5) $$P = 4\iint_D \iint_D g(z,\zeta)\,dx\,dy\,d\xi\,d\eta \quad .$$

This formula is often a convenient tool for dealing with torsional rigidity. It permits computation of P by quadratures if the conformal mapping of D onto the unit circle is known. In fact, let

(1.6) $$z = F(w)$$

be the function which is univalent and analytic in $|w| < 1$ and maps this domain onto D. If $\zeta = F(\omega)$, we have clearly (transplantation of the Green's function)

(1.7) $$g(z,\zeta) = \frac{1}{2\pi} \log\left|\frac{1-\bar{\omega}w}{w-\omega}\right|$$

and we can bring the formula (1.5) into the form

(1.8) $$P = \frac{2}{\pi} \iint_{|w|<1} \iint_{|\omega|<1} \log\left|\frac{1-\bar{\omega}w}{w-\omega}\right| |F'(w)|^2 |F'(\omega)|^2\,du\,dv\,d\rho\,d\sigma$$

if $w = u + iv$, $\omega = \rho + i\sigma$. From (1.8), we could derive a formula for P in terms of the coefficients of the Taylor series for F(w) around the origin [9, 15].

1.3 As another application, we deal with the following situation. Let D lie in a larger domain D_1 whose Green's function and torsional rigidity we denote by $g_1(z,\zeta)$ and P_1, respectively. From the maximum definition for the torsional rigidity (I, 1.5), we deduce

$$P_1 \geq P \tag{1.9}$$

and, from the maximum principle for harmonic functions, we derive easily

$$g_1(z,\zeta) \geq g(z,\zeta) \quad . \tag{1.10}$$

Obviously, (1.9) is a consequence of (1.10) and the identity (1.5).

We can obtain a better upper bound for P than (1.9) in the form

$$P \leq 4 \iint_D \iint_D g_1(z,\zeta)\, dx\, dy\, d\xi\, d\eta \quad . \tag{1.11}$$

This formula is often useful if the Green's function of the domain D is not available or is very complicated while the Green's function of a slightly larger domain is easier to handle.

1.4. We consider now the sequence of domains D(t) which are obtained from the given domain $D = D(0)$ by the conformal mappings (1.1). Let $g_t(z,\zeta)$ be the Green's function of D(t) and P(t) its torsional rigidity. We have

$$g_t(z + tf(z), \zeta + tf(\zeta)) = g(z,\zeta) \quad . \tag{1.12}$$

where $g(z,\zeta)$ is the Green's function of D. In fact, both sides of (1.12) are harmonic in D and satisfy the requirements a) - c) of Section 1.2, which determine the Green's function in a unique way.

We apply the identity (1.5) for the domain D(t) and its corresponding functionals; we then refer back to D(0) by means of (1.1) and (1.12). In this way, we obtain

$$P(t) = 4 \iint_D \iint_D g(z,\zeta) |1 + tf'(z)|^2 \cdot |1 + tf'(\zeta)|^2\, dx\, dy\, d\xi\, d\eta \quad . \tag{1.13}$$

This formula shows that P(t) is a polynomial of the fourth degree in t.

We calculate from (1.13):

$$(1.14)\qquad \frac{d^2}{dt^2}P(t) = 8\iint_D\iint_D g(z,\zeta)(|f'(z)|^2|1+tf'(\zeta)|^2$$

$$+|f'(\zeta)|^2|1+tf'(z)|^2)dx\,dy\,d\xi\,d\eta + 32\iint_D\iint_D g(z,\zeta)(\mathrm{Re}\{f'(z)\}$$

$$+t|f'(z)|^2)(\mathrm{Re}\{f'(\zeta)\}+t|f'(\zeta)|^2)dx\,dy\,d\xi\,d\eta \qquad .$$

It is a well-known fact that the Green's function is positive-definite; that is,

$$(1.15)\qquad \iint_D\iint_D g(z,\zeta)\ell(z)\ell(\zeta)dx\,dy\,d\xi\,d\eta > 0$$

for every continuous, not identically vanishing, real function $\ell(z)$. Hence, we have

$$(1.16)\qquad P''(t) > 0$$

except for the case that $f'(z)\equiv 0$. Thus, we have proved:

Theorem 1. If the domains $D(t)$ are obtained from a given domain $D(0)$ by the conformal transformations (1.1), their torsional rigidity $P(t)$ is convex from below as a function of the parameter t.

This convexity statement could have been obtained also from the maximum definition (I, 1.5) by transplantation of the extremal function $f(x,y)$ and by using the invariance of the Dirichlet integral under conformal mapping. The above derivation is perhaps more illuminating since it shows the explicit dependence of $P(t)$ upon the parameter.

1.2. We make the following application of Theorem 1. Let D be the unit circle $|z|<1$ and consider conformal transformations (1.1) with $f(0)=f'(0)=0$. We derive from (1.13) for all such transformations

$$(1.17)\qquad P'(0) = 8\iint_D\iint_D g(z,\zeta)[\mathrm{Re}\{f'(z)\}+\mathrm{Re}\{f'(\zeta)\}]dx\,dy\,d\xi\,d\eta$$

$$= 16\iint_D\iint_D g(z,\zeta)\mathrm{Re}\{f'(\zeta)\}dx\,dy\,d\xi\,d\eta \qquad ,$$

by virtue of the symmetry of the Green's function. We see immediately that

(1.18) $$\iint_D g(z,\zeta)\mathrm{Re}\{f'(\zeta)\}\,d\xi\,d\eta = -\frac{1}{4}\mathrm{Re}\left\{\bar{z}f(z)-\frac{1}{z}f(z)\right\} \quad ;$$

in fact, by the Laplace-Poisson equation, we recognize that both sides have the same Laplacian $-\mathrm{Re}\{f'(z)\}$, and since $\bar{z} = 1/z$ on the boundary of the unit circle, both do vanish there. But these two facts clearly imply the identity of both sides.

We observe next that in view of our restrictions regarding $f(z)$, we have

(1.19) $$\mathrm{Re}\left\{\bar{z}f(z)-\frac{1}{z}f(z)\right\} = \sum_{\nu=1}^{\infty}[a_\nu(r)\cos\nu\varphi + b_\nu(r)\sin\nu\varphi] \quad .$$

Hence, clearly we obtain from (1.17)

(1.20) $$P'(0) = 0 \quad .$$

Because of the convexity of $P(t)$ from below, we have

(1.21) $$P'(t) > 0 \quad \text{for} \quad t > 0$$

and

(1.22) $$P(t) > P(0) \quad .$$

Using the same considerations as in Section III, 3.3, we obtain

Theorem 2. Let r_i be the inner radius of a simply-connected finite domain and P its torsional rigidity. Then Pr_i^{-4} has the minimum value if D is a circle.

Theorem 2 expresses the inequality

(1.23) $$P \geqslant \frac{\pi}{2} r_i^4$$

valid for all simply-connected domains D [15].

2. Further Applications.

2.1. Various other functionals are connected with the Green's function $g(z,\zeta)$ of a domain D. We mention, as an example, the iterated Green's function

(2.1) $$g^{(2)}(z,\zeta) = \iint_D g(z,w)g(w,\zeta)\,du\,dv \quad , \quad w = u + iv \quad .$$

This function is the Green's function for the biharmonic differential equation

(2.2) $\qquad \Delta\Delta u = 0$

with the boundary conditions

(2.3) $\qquad u = \Lambda u = 0 \quad$ on C.

It plays a role in the elasticity theory of plane plates with the boundary curve C. It is easily seen that $g^{(2)}(z,\zeta)$ is positive and continuous in D.

Suppose now that we transform $D = D(0)$ by conformal mappings (1.1) into a sequence of domains $D(t)$. Let $g_t^{(2)}(z,\zeta)$ be the corresponding iterated Green's functions. Using the definition (2.1) with respect to the domain $D(t)$ and referring back to the original domain D by means of (1.1) and (1.12), we find:

(2.4) $$g_t^{(2)}(z^*,\zeta^*) = \iint_D g(z,w)g(w,\zeta)|1 + tf'(w)|^2\,du\,dv \quad .$$

Let us choose an arbitrary but fixed pair of points z,ζ in D. The functional

(2.5) $$G(t) = g_t^{(2)}(z + tf(z),\, \zeta + tf(\zeta))$$

is the value of $g_t^{(2)}$ at the transplanted points z^*,ζ^*. We see from (2.4) that $G(t)$ is a second degree polynomial in t. We calculate immediately:

(2.6) $$G''(t) = G''(0) = 2\iint_D g(z,w)g(w,\zeta)|f'(w)|^2\,du\,dv > 0$$

(2.7) $$G'(t) = 2\iint_D g(z,w)g(w,\zeta)\mathrm{Re}\left\{f'(w)\right\}du\,dv + tG''(0) \quad .$$

We see that G(t) is convex from below in dependence on t.

2.2. In order to apply our preceding results, we choose again D to be the unit circle $|z| < 1$ and restrict ourselves to functions $f(z)$ which are analytic in the unit circle and satisfy $f(0) = f'(0) = 0$. We specialize further the pair of points z,ζ by putting $z = \zeta = 0$. Observe that $z^* = \zeta^* = 0$

for all mappings (1.1) with the above functions $f(z)$.

We have in the case of the unit circle

(2.8) $$g(w,0) = -\frac{1}{2\pi}\log|w|$$

and hence

(2.9) $$G'(0) = \frac{1}{2\pi^2}\iint_D (\log|w|)^2 \operatorname{Re}\{f'(w)\}\,du\,dv \quad .$$

In view of the radial symmetry of the Green's function $g(w,0)$ and of the normalization of $f(w)$, we recognize that

(2.10) $$G'(0) = 0 \quad .$$

Hence, the convexity of $G(t)$ implies that

(2.11) $$G(t) \geqslant G(0)$$

and equality is only possible if $f'(z) \equiv 0$, i.e., if $D(t) = D(0)$.

In the usual way, we may deduce from (2.11) the following

Theorem 3. If r_i is the inner radius of D and $g^{(2)}(z,\zeta)$ is the iterated Green's function of the domain, the expression $r_i^{-2}g^{(2)}(z,z)$ has a minimum if D is a circle with z as center.

It is easy to calculate $g^{(2)}(0,0)$ for the unit circle; we find for its value $(8\pi)^{-1}$ and may, therefore, express Theorem 3 by the inequality

(2.12) $$g^{(2)}(z,z) \geqslant \frac{1}{8\pi}\, r_i^2 \quad .$$

APPENDIX

by

Heinz Helferstein

I. On the Torsional Rigidity of an Isosceles Triangle.

1. Let a denote the base, h the height, and A the area of an isosceles triangle whose torsional rigidity is P. If we keep $a = 2$ fixed, the shape of the triangle is determined by the variable quantity h. We can determine the value of P in the following special cases:

infinitely acute triangle:	$h \to \infty$,	$P \sim \frac{2}{3} \cdot h$;
equilateral triangle:	$h = \sqrt{3}$,	$P = \sqrt{3}/5$;
right isosceles triangle:	$h = 1$,	$P = 0.104364$;
infinitely flat triangle:	$h \to 0$,	$P \sim h^3/6$.

The results for $h = \sqrt{3}$ and $h = 1$ are well known [15, p. 256, 257]. The result for $h \to \infty$ easily follows from the value of P for an infinitely narrow sector [15, p. 256]. The result for $h \to 0$ can be obtained by two symmetrizations. In fact, for a right triangle with legs 2 and h, P can be estimated from the P of a narrow sector when $h \to 0$. Symmetrization with respect to a line parallel to the short leg h changes this right triangle into a (flat) isosceles triangle, which is changed by a second symmetrization, with respect to a line parallel to the base, into a rhombus, the torsional rigidity of which can be estimated from above by use of an inequality due to E. Nicolai [15, p. 112(1)]. In view of the fact that the symmetrization increases P, the torsional rigidities of the three figures considered turn out to be equal as $h \to 0$.

2. We shall use four inequalities for the P of our isosceles triangle [12]. If we introduce the notation

$$p = 2(3 + h^2) \quad , \quad q = (1 + h^2)(9 + h^2) \quad , \quad r = 4(1 + h^2)^2 \quad ,$$

then

$$\frac{A^2}{5\sqrt{3}} \geqslant P \geqslant \frac{4A^3}{5p} \quad ;$$

(A.1)

$$\frac{4A^3(43p^2 - 93q)}{3(43p^3 - 93pq + 216r)} \geqslant P \geqslant \frac{28A^3(11p^2 + 16q)}{45(5p^3 + 16pq + 64r)} \quad .$$

The first (simpler) upper bound for P expresses the fact that of all triangles with a given area A the equilateral triangle has the largest torsional rigidity [15, p. 158]. The second upper bound for P is based on Thomson's principle; details of the derivation will be published elsewhere. The first lower bound for P is a consequence of the convexity of the quantity $\omega = h/P$, considered as a function of $\tau = 1/h^2$ (see Section I, 1.1) and of the slope of the curve $\omega(\tau)$ at $\tau = 3^{-1/2}$ given in (IV, 4.3). The second lower bound is discussed in a previous report [13, p. 22, formula (6.18)].

3. We can derive closer bounds for P if we combine the above inequalities with the fact that $\omega = h/P$ is a convex and increasing function of $\tau = 1/h^2$.

The four triangles of which the torsional rigidity can be precisely computed, yield four points:

$$\tau = 0 \;, \quad \omega = 3/2 \quad ;$$

$$\tau = 1/3, \quad \omega = 5 \quad ;$$

$$\tau = 1 \;, \quad \omega = 9.5821 \quad ;$$

$$\lim_{\tau \to \infty} \frac{\omega}{\tau} = 6 \quad .$$

The last relation yields the slope of the asymptote of the curve $\omega(\tau)$. The inequalities (A.1) can easily be translated into the (ω, τ)-notation. We write

$$\omega_1 = \frac{5}{2}(3\tau + 1) \quad ,$$

$$\omega_2 = 5\sqrt{3\tau} \quad ,$$

$$\omega_3 = \frac{45}{14}\,\frac{(11\tau + 1)(5\tau + 3)^2}{135\tau^2 + 196\tau + 15} \quad ,$$

$$\omega_4 = \frac{3([illegible] + [illegible]\tau + 1881\tau^2 + 2565\tau^3)}{2([illegible] + 1[illegible]\tau + 711\tau^2)} \quad .$$

Then, ω_1 and ω_3 are upper bounds, ω_2 and ω_4 lower bounds for $\omega(\tau)$. In a (τ,ω)-diagram, ω_1 represents the common tangent of ω_2, ω_3, and ω_4 at the point $(1/3,5)$. The expression ω_3 is in other points always less than ω_1 (this follows also from the derivation of these quantities) and so, in looking for an upper bound, we always prefer ω_3 to ω_1. By numerical computation it is found that ω_2 and ω_4 intersect (besides at $(1/3,5)$ for $\tau_0 = 0.12145$, and we have $\omega_4 > \omega_2$ for $0 \leq \tau < \tau_0$, $\omega_2 \geq \omega_4$ for $\tau > \tau_0$. The curve ω_4 passes through the correct points $(0,3/2)$ and $(1/3,5)$. From the convexity we conclude that the straight line that joins these points and may be called ω_5 is another lower bound between them, but ω_4 and ω_2 are better ones. The same is true in comparison with the tangent $\omega = 12.5\tau + 1.5$ from the point $(0,3/2)$ to ω_2, since the tangent of ω_4 has a greater slope in this point (viz. 12.70). ω_2 and ω_4 have a common tangent which is, by convexity, the best lower bound for ω in the neighborhood of their intersection. However, since these curves intersect under a very small angle the improvement is very unimportant and valid for a very small range of values of τ only.

The line joining the precisely known points $(1/3,5)$ and $(1, 9.5821)$ is again a lower bound between these points from convexity. But it can be improved by drawing the tangent

$$\omega_7 = 6.8413\tau + 2.7409$$

from (1, 9.5821) to ω_2 which touches the latter curve at $\tau = 0.4006$. The best lower bounds in this interval are therefore ω_2 and ω_7.

Since ω_3 does not pass through the correct point (1, 9.5821) but a little higher (1, 9.6429) we can, in the left-hand neighborhood of this point, improve the upper bound ω_3 by drawing through (1, 9.5821) the line parallel to the asymptote of $\omega(\tau)$. Since we know its slope to be 6, its equation can be found

$$\omega_6 = 6\tau + 3.5821 \quad .$$

ω_3 and ω_6 intersect for $\tau = 0.9183$. By convexity ω_7 is the best upper bound of ω on the right-hand side of the point (1, 9.5821), as long as it is less than ω_3. ω_3 and ω_7 intersect for $\tau = 1.3884$. The best lower bound for ω on the right-hand side of 1 is ω_6, since ω_2 and ω_4 are both less than ω_6. For large values of τ the curve ω_3 can be replaced by its asymptote $\omega = 6.5476\,\tau + 3.3113$.

Summing up we have found the following bounds:

$0 \leq \tau \leq 0.12145$	:	$\omega_4 \leq \omega < \omega_3$,
$0.12145 \leq \tau \leq 0.40064$		$\omega_2 \leq \omega \leq \omega_3$,
$0.40064 \leq \tau \leq 0.91827$		$\omega_7 < \omega < \omega_3$,
$0.91827 \leq \tau \leq 1$		$\omega_7 \leq \omega \leq \omega_6$.
$1 \leq \tau \leq 1.38840$		$\omega_6 \leq \omega \leq \omega_7$,
$1.38840 \leq \tau$		$\omega_6 < \omega < \omega_3$.

Figure II represents the points (0, 3/2), (1/3, 5) and (1, 9.5821); the shape of the corresponding triangle is indicated near to the baseline. Essential portions of the straight lines ω_1, ω_6, and ω_7 and of the parabola ω_2 are also given. Yet of ω_3, ω_4, and ω_5 only the points of intersection with the left or right border of the figure are indicated.

We introduce the quantities $q = h/a = \frac{1}{2\sqrt{\tau}}$ and $Q = \frac{100\,P}{a^4} = \frac{25}{4\sqrt{\tau}\,\omega}$ and translate the above inequalities which yield:

$$q \geq 1.435 \quad : \quad \frac{12.5q}{\omega_3} < Q \leq \frac{12.5q}{\omega_4} \quad ,$$

$$1.435 \geq q \geq 0.791 \quad : \quad \frac{12.5q}{\omega_3} \leq Q \leq \frac{12.5q}{\omega_2} \quad ,$$

$$0.791 \geq q \geq 0.526 \quad : \quad \frac{12.5q}{\omega_3} < Q < \frac{12.5q}{\omega_7} \quad ,$$

$$0.526 \geq q \geq 0.500 \quad : \quad \frac{12.5q}{\omega_6} \leq Q \leq \frac{12.5q}{\omega_7} \quad ,$$

$$0.500 \geq q \geq 0.424 \quad : \quad \frac{12.5q}{\omega_7} \leq Q \leq \frac{12.5q}{\omega_6} \quad ,$$

$$0.424 \geq q \quad : \quad \frac{12.5q}{\omega_3} < Q < \frac{12.5q}{\omega_6} \quad .$$

By means of these inequalities the Table I has been prepared. It is remarkable that for all values of q the relative error never surpasses 12.5%, and that for $q < 2$ it is even less than 6%. In the neighborhood of the precisely known cases it drops down to much smaller values. In about two thirds of the cases listed in Table I the relative error is under 1%.

TABLE I. Torsional rigidity of an isosceles triangle.

P is the torsional rigidity of an isosceles triangle with base a and height h. The columns list:

(1) $\tau = a^2/4h^2$

(2) h/a

(3) lower bound ℓ for $100\ P/a^4$

(4) upper bound u for $100\ P/a^4$

(5) approximate value $2\ell u/(\ell + u)$ for $100\ P/a^4$

(6) extreme possible relative error $100(u-\ell)/(u+\ell)$, computed in percentages, of the approximation given in column (5).

(1)	(2)	(3)	(4)	(5)	(6)
τ	h/a	lower	upper	100 P/a^4	error %
0.00	∞	$\sim$ 6.4815 h/a	$\sim$ 8.33 h/a	$\sim$ 7.2917 h/a	12.50
0.05	2.2360	10.9615	13.0553	11.9171	8.72
0.10	1.5811	6.4417	7.1503	6.7775	5.21
0.15	1.2910	4.5722	4.8113	4.6887	2.55
0.20	1.1180	3.5327	3.6081	3.5702	1.06
0.25	1.0000	2.8671	2.8868	2.8769	0.34
0.30	0.9129	2.4034	2.4056	2.4045	0.05
0.35	0.8452	2.0616	2.0620	2.0618	0.01
0.40	0.7906	1.7991	1.8042	1.8016	0.01
0.45	0.7454	1.5914	1.6010	1.5962	0.30
0.50	0.7071	1.4230	1.4345	1.4287	0.40
0.55	0.6742	1.2839	1.2958	1.2898	0.46
0.60	0.6455	1.1670	1.1787	1.1728	0.50
0.65	0.6202	1.0677	1.0785	1.0731	0.51
0.70	0.5976	0.9653	0.9921	0.9785	1.37
0.75	0.5774	0.9080	0.9168	0.9124	0.48
0.80	0.5590	0.8430	0.8507	0.8468	0.46
0.85	0.5423	0.7856	0.7923	0.7889	0.43
0.90	0.5270	0.7346	0.7404	0.7375	0.39
0.95	0.5130	0.6908	0.6940	0.6924	0.23
1.00	0.5000	0.6523	0.6523	0.6523	0.00
1.05	0.4880	0.6146	0.6172	0.6159	0.21
1.10	0.4767	0.5805	0.5853	0.5828	0.41
1.15	0.4662	0.5494	0.5560	0.5527	0.60
1.20	0.4564	0.5210	0.5292	0.5251	0.77
1.25	0.4472	0.4950	0.5044	0.4997	0.94
1.30	0.4385	0.4712	0.4816	0.4763	1.10
1.35	0.4303	0.4491	0.4605	0.4547	1.24
1.40	0.4226	0.4289	0.4408	0.4348	1.38
1.50	0.4082	0.3931	0.4056	0.3993	1.56
1.60	0.3953	0.3621	0.3748	0.3684	1.72
1.70	0.3835	0.3351	0.3478	0.3413	1.87
∞	0	$\sim \frac{100}{a^4}\,\frac{21ah^3}{275}$	$\sim \frac{100}{a^4}\,\frac{ah^3}{12}$	---	4.36

II. On the Torsional Rigidity of a Rectangle.

1. Let a, b be the sides of a rectangle, $q = a/b$ their ratio, and P its torsional rigidity. We keep $b = 1$ fixed and let a vary. P is known in the case of the square ($a = 1$), namely $P_o = 0.1406$.

Moreover, the following bounds will be used:

$$P \leq \frac{1}{3}\ \frac{a^3 b^3}{a^2 + b^2}$$

which goes over in an equation both for $a/b \rightarrow 0$ and for $a/b \rightarrow \infty$,

$$P \geqslant P_o \frac{2a^3 b^3}{a^2 + b^2} \qquad ,$$

$$P \leq P_o a^2 b^2 \qquad .$$

The last inequality follows from the fact that of all quadrilaterals with a given area the square has the highest torsional rigidity [15, p. 158]. The first inequality (an upper bound for P) is a particular case of an inequality due to E. Nicolai [15, p. 112 (1)]. The middle inequality (a lower bound for P) is a particular case of (IV, 4.28).

2. Similarly as in Section A.I.3 the quantity $a/P = \omega$ or $\Omega = P_o\omega = \frac{aP_o}{P}$ is a convex function of the variable $\frac{1}{a^2} = \tau$.

We define the following functions:

$$\Omega_1 = 3P_o(1+\tau) \quad , \quad \Omega_2 = \frac{1}{2}(1+\tau) \quad , \quad \Omega_3 = \sqrt{\tau} \qquad .$$

Then the above inequalities take the form

$$\Omega \geqslant \Omega_1 \quad , \ \Omega \leq \Omega_2 \quad , \ \Omega \geqslant \Omega_3 \qquad ,$$

and the precisely known cases are expressed by the equations

$$\Omega = 3P_o \quad \text{for} \quad \tau = 0 \quad , \quad \text{and} \ \Omega = 1 \quad \text{for} \quad \tau = 1 \qquad .$$

In a τ,Ω diagram Ω_2 is the tangent of the parabola Ω_3 at the point (1,1). From the convexity of $\Omega(\tau)$ we conclude that the tangent from the correct point $(0, 3P_o)$ to Ω_3 is a better lower bound for Ω

than Ω_3. The equation of this tangent is

$$\Omega_4 = \frac{1}{12P_o}\tau + 3P_o = 0.5927\tau + 0.4218 \quad .$$

This tangent is even a better bound than the line joining the two known points $(0, 3P_o)$, and $(1,1)$, since the slope of the latter is only $1-3P_o = 0.5782$. The point of contact of Ω_3 and Ω_4 is $\tau = 0.7117$, $\Omega = 0.8436$.

Likewise the tangent to Ω_3 parallel to Ω_1 is a better lower bound for Ω than Ω_1. Its equation is

$$\Omega_5 = 3P_o\cdot\tau + \frac{1}{12P_o} = 0.4218\tau + 0.5927 \quad ,$$

and its point of contact $\tau = 1.4052$, $\Omega = 1.1854$.

Using the convexity, the best available bounds for Ω are therefore found to be:

$$\begin{aligned} 0 \leq \tau \leq 0.7117 \quad &: \quad \Omega_4 \leq \Omega \leq \Omega_2 \quad , \\ 0.7117 \leq \tau \leq 1.4052 \quad &: \quad \Omega_3 \leq \Omega \leq \Omega_2 \quad , \\ 1.4052 \leq \tau \quad &: \quad \Omega_5 \leq \Omega \leq \Omega_2 \quad . \end{aligned}$$

If in these inequalities we introduce the quantities $q = a/b$, $Q = P/ab^3$, instead of τ, Ω, they take the form:

$$\begin{aligned} q \geq 1.1854 \quad &: \quad \frac{2P_o q^2}{q^2+1} \leq Q \leq \frac{12P_o^2 q^2}{36P_o^2 q^2 + 1} \quad . \\ 1.1854 \geq q \geq 0.8436 \quad &: \quad \frac{2P_o q^2}{q^2+1} \leq Q \leq P_o q \quad , \\ 0.8436 \geq q \quad &: \quad \frac{2P_o q^2}{q^2+1} \leq Q \leq \frac{12P_o^2 q^2}{36P_o^2 + q^2} \quad . \end{aligned}$$

Column (4) of Table II contains the best approximation to the torsional rigidity P of a rectangle that we can compute on the basis of the inequalities listed. Column (6) of the same table shows the extreme possible error (in

percentages) of the approximate value listed in column (4); this error is estimated *a priori*, without reference to the exact value, only on the basis of the lower and upper estimates listed in columns (2) and (3), respectively. The actual deviation of the approximate value in column (4) from the actual value in column (5) is much less in most cases than the extreme error given in column (6).

3. Here is a remark that may considerably enhance the interest of Table II.

Let P' denote the torsional rigidity of a rhombus with diagonals a and b. Then the following inequalities hold

$$P' \leq \frac{1}{12} \frac{a^3 b^3}{a^2 + b^2}$$

which goes over in an equation both for $a/b \to 0$ and for $a/b \to \infty$,

$$P' \geq \frac{P_o}{2} \frac{a^3 b^3}{a^2 + b^2} \quad .$$

$$P' \leq P_o a^3 b^3 / 4 \quad ;$$

$P_o = 0.1406$ is the torsional rigidity of the unit square. These three inequalities are derived from the same sources as the corresponding three inequalities for P listed in Section 1, and we can summarize them by saying that the same three inequalities hold for $4P'$ as for P. Therefore, the columns (2), (3), (4), and (6) of Table II yield lower and upper bounds, an approximate value, and an estimate of the extreme possible error of this approximate value for $4P'$ just as well as for P.

Now, $4P'$ precisely coincides with P for $a = b$, $a/b \to 0$, $a/b \to \infty$, yet we do not know and can scarcely expect that a precise coincidence takes place for any other value of the ratio a/b. However, it seem reasonable to take the (rather well known) value $P/4$ as an approximation for the (unknown) P'. Table II could be used to estimate the error of this approximation numerically.

Table II. Torsional Rigidity of a Rectangle.

P is the torsional rigidity of a rectangle with sides a and b, $a \geq b$. The columns list:

(1) q - a/b ratio of longer side a to shorter side b.

(2) lower bound ℓ for P/ab^3,

(3) upper bound u for P/ab^3,

(4) approximate value $2\ell u/(\ell+u)$ for P/ab^3 ,

(5) exact value for P/ab^3,

(6) extreme possible relative error $100(u-\ell)/(u+\ell)$, computed in percentages, of the approximation given in column (4).

(1) a/b	(2) lower	(3) upper	(4) $Pa^{-1}b^{-3}$ prox.	(5) $Pa^{-1}b^{-3}$ ex.	(6) error %
1	0.1406	0.1406	0.1406	0.1406	0.00
1.1	0.1540	0.1547	0.1543	0.1540	0.23
1.2	0.1660	0.1687	0.1673	0.166	0.82
1.25	0.1715	0.1755	0.1735	0.1717	1.16
1.5	0.1947	0.2052	0.1998	0.1958	2.63
1.75	0.2120	0.2285	0.2199	0.2143	3.75
2	0.2250	0.2467	0.2353	0.2287	4.60
2.5	0.2424	0.2722	0.2564	0.2494	5.78
3	0.2531	0.2883	0.2696	0.2633	6.51
4	0.2647	0.3064	0.2840	0.2808	7.31
5	0.2704	0.3156	0.2912	0.2913	7.71
6	0.2736	0.3208	0.2953	0.2983	7.94
7	0.2756	0.3240	0.2978	0.3033	8.08
8	0.2769	0.3262	0.2995	0.3071	8.17
10	0.2784	0.3287	0.3015	0.3123	8.28
12	0.2793	0.3301	0.3026	0.3158	8.34
100	0.2812	0.3333	0.3050	0.3312	8.48
∞	0.2812	0.3333	0.3051	0.3333	8.48

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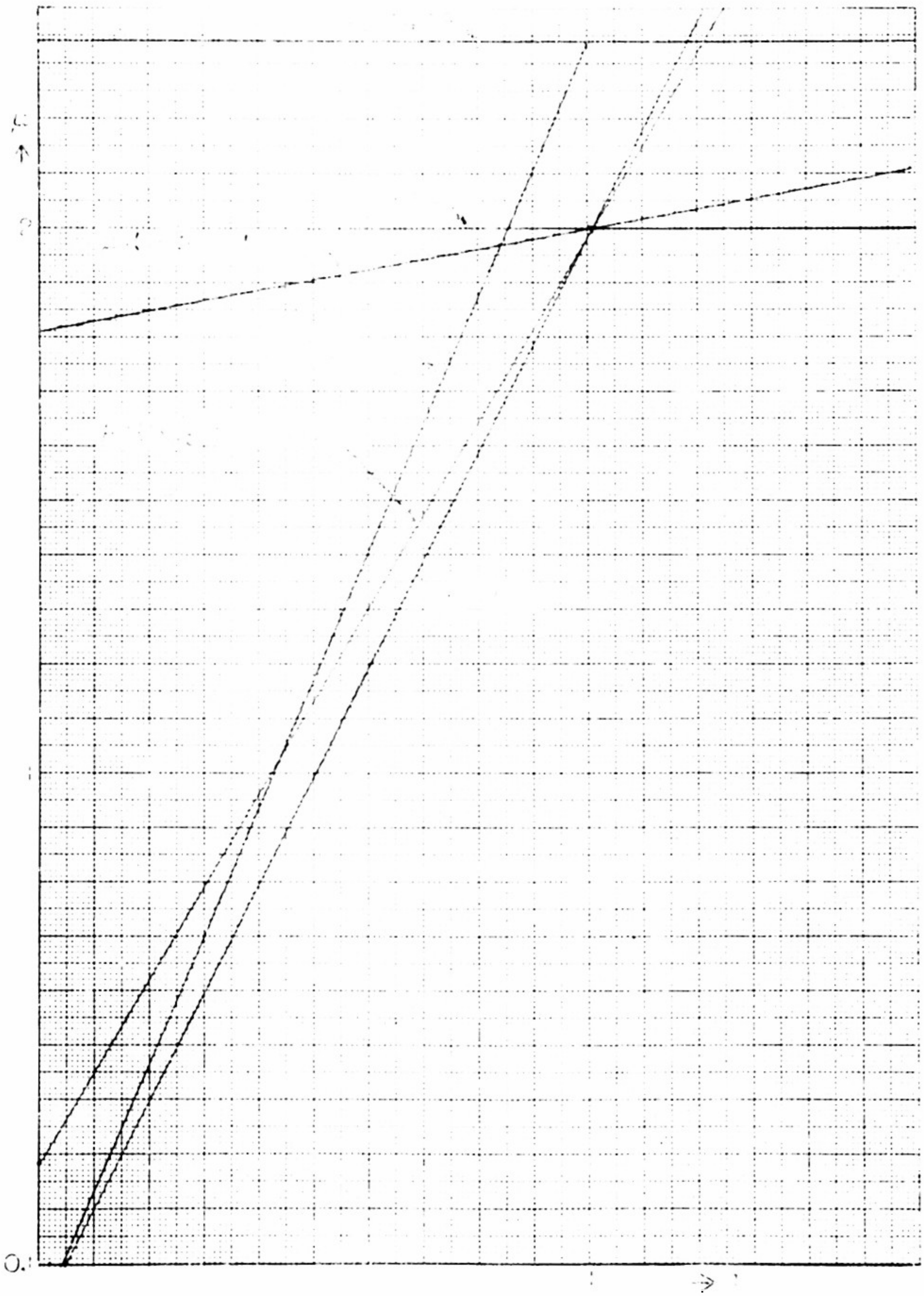

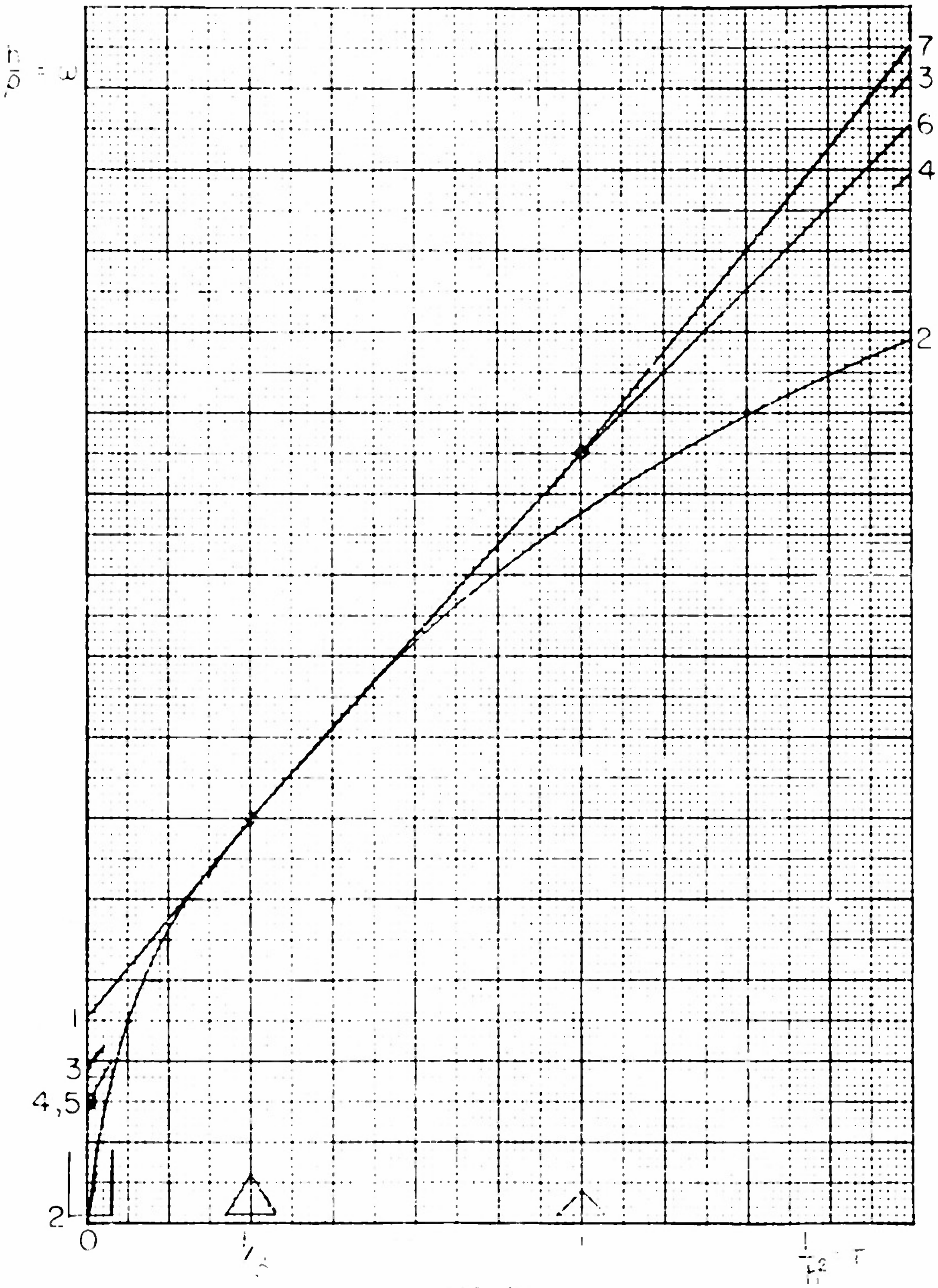

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